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q-Classical polynomials and the q-Askey and Nikiforov–Uvarov tableaus

R. Álvarez-Nodarse[a,b], J.C. Medem[a,*]

[a] Departamento de Análisis Matemático, Universidad de Sevilla, Apdo. 1160, E-41080 Sevilla, Spain

[b] Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, E-18071 Granada, Spain

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Abstract

In this paper we continue the study of the q-classical (discrete) polynomials (in the Hahn's sense) started in Medem et al. (this issue, Comput. Appl. Math. 135 (2001) 157–196). Here we will compare our scheme with the well known q-Askey scheme and the Nikiforov–Uvarov tableau. Also, new families of q-polynomials are introduced.

Keywords: Orthogonal polynomials; Exponential lattices; q-Difference equations; q-Rodrigues formula

0. Introduction

The so-called q-polynomials constitute a very important and interesting set of special functions and more specifically of orthogonal polynomials. They appear in several branches of the natural sciences, e.g., continued fractions, Eulerian series, theta functions, elliptic functions, …; see [3,9], quantum groups and algebras [14,15,25], among others (see also [10,20]). They have been intensively studied in the last years by several people (see, e.g., [13]) using different tools. One of them is reviewed in [13] which is based on the basic hypergeometric series [10] and was developed mainly by the American School starting from the works of Andrews and Askey (see, e.g., [4]). The literature on this method is so vast that we are not able to include it here. A very complete list is given in [13] and leads to the so-called q-Askey tableau of hypergeometric polynomials [13]. In another direction, the Russian (former Soviet) school, starting from the works in [21] and further developed

* Corresponding author. Tel.: +34-954-55-7997; fax: 954-55-7972.

E-mail addresses: ran@cica.es (R. Álvarez-Nodarse), jcmedem@cica.es (J.C. Medem).

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by Atakishiyev and Suslov (see, e.g., [5,6,20,23,24] and references contained therein), has considered the hypergeometric difference equation on nonuniform lattices [22], from where the hypergeometric representation of the q-polynomials follows in a very simple way [5,23]. This schema leads to the Nikiforov–Uvarov tableau [20,23] for the polynomial solutions of the difference hypergeometric equation on nonuniform lattices. The paper by Atakishiyev and Suslov [6] deserves a special mention where a difference analog of the well-known method of indeterminate coefficients has been developed for the hypergeometric equation on nonuniform lattices and also give a classification similar to the Nikiforov and Uvarov 1991 one but for the q-special functions (not only for the polynomial solutions).

Our main aims here are two: to continue the study started in [18] using the algebraic theory developed in [16] and to classify the q-classical polynomials and compare them with the q-Askey and Nikiforov–Uvarov Tableaus. In fact, in [18] we have proven several characterizations of the q-classical polynomials as well as a very simple computational algorithm for finding their main characteristics (e.g., the coefficients of the three-term recurrence relation, structure relation of Al-Salam, Chihara, etc). Going further, we will give here a "very natural" classification of the q-classical polynomials introduced by Hahn in his paper [11], i.e., we will classify all orthogonal polynomial sequences such that their q-differences, defined by $\Theta f(x) = (f(qx) - f(x))/(q-1)x$ are orthogonal in the widespread sense: the q-Hahn tableau (a first step on this in the framework of the q-Askey tableau was done in [15]). Notice that the aforesaid polynomials are instances of the q-polynomials on the linear exponential lattice $x(s) = c_1 q^s$. For several surveys about this lattice and their corresponding polynomials see [2,4,5,8,10,13,20,24] as well as Section 3.2 from below. Furthermore, we will compare our classification (q-classical tableau) with the aforesaid two schemes. From this comparison we find that there are two missing families in the q-Askey schema (one of them is a nonpositive-definite family) and using the results of [18] we study them with detail. Also, the correspondence of this q-classical schema and the Nikiforov and Uvarov one will be established. In such a way a complete correspondence between the q-classical families of the q-Askey and Nikiforov and Uvarov tableaus for exponential linear lattices will be shown.

In that way we will give here a natural continuation of the papers [11,5,18,23] providing a careful discussion and analysis of all possible solutions of the Pearson-type equation on the linear exponential lattice, obtaining, explicitly, the corresponding weight functions in terms of the polynomials coefficients of the Pearson-type equation (Theorems 2.1 and 2.2) and consequently of the difference equation of hypergeometric type. Let us point out that Theorem 2.1 was proven by Hahn using a completely different procedure. The method used here is a straightforward generalization of an idea proposed in [12] and is new as far as we know. Finally, we present a new schema for the q-orthogonal polynomials on the exponential lattice (an analog of the so-called q-Hahn tableau [15]) and compare it with the q-Askey and Nikiforov–Uvarov tableaus. This comparison allow us to obtain two new families of q-polynomials not considered in the literature.

The structure of the paper is as follows. In Section 1 we introduce some notations and definitions useful for the subsequent ones. In Section 2, the q-weight functions are introduced and computed for all q-classical families. This will allow to classify all orthogonal polynomial families of the q-Hahn tableau. Finally, in Section 3, several applications are considered: the classification of the q-classical polynomials (q-Hahn tableau), the integral representation for the orthogonality, the explicit expression, the hypergeometric representation of these q-classical polynomials as well as the detailed study of two new families of q-polynomials.

1. Preliminaries

In this section we will give a brief survey of the operational calculus and some basic concepts and results needed for the rest of the work.

Let $\mathbb{P}$ be the linear space of polynomial functions in $\mathbb{C}$ with complex coefficients and $\mathbb{P}^*$ be its algebraic dual space, i.e., $\mathbb{P}^*$ is the linear space of all linear applications $\boldsymbol{u}:\mathbb{P}\to\mathbb{C}$. In the following, we will refer to the elements of $\mathbb{P}^*$ as functionals and we will denote them with bold letters $(\boldsymbol{u},\boldsymbol{v},\ldots)$.

Since the elements of $\mathbb{P}^*$ are linear functionals, it is possible to determine them from their actions on a given basis $(B_n)_{n\geqslant 0}$ of $\mathbb{P}$, e.g., the canonical basis of $\mathbb{P}$, $(x^n)_{n\geqslant 0}$. In general, we will represent the action of a functional over a polynomial by $\langle \boldsymbol{u},\pi\rangle$, $\boldsymbol{u}\in\mathbb{P}^*$, $\pi\in\mathbb{P}$, and therefore a functional is completely determined by a sequence of complex numbers $\langle \boldsymbol{u},x^n\rangle = u_n$, $n\geqslant 0$, the so-called moments of the functional.

Definition 1.1. Let $(P_n)_{n\geqslant 0}$ be a basis sequence of $\mathbb{P}$ such that $\deg P_n = n$. We say that $(P_n)_{n\geqslant 0}$ is an orthogonal polynomial sequence (OPS in short), if and only if there exists a functional $\boldsymbol{u}\in\mathbb{P}^*$ such that $\langle \boldsymbol{u},P_mP_n\rangle = k_n\delta_{mn}$, $k_n\neq 0$, $n\geqslant 0$, where δ_{mn} is the Kronecker delta. If $k_n>0$ for all $n\geqslant 0$, we say that $(P_n)_{n\geqslant 0}$ is a positive definite OPS.

Definition 1.2. Let $\boldsymbol{u}\in\mathbb{P}^*$ be a functional. We say that $\boldsymbol{u}$ is a quasi-definite functional if and only if there exists a polynomial sequence $(P_n)_{n\geqslant 0}$, which is orthogonal with respect to $\boldsymbol{u}$. If $(P_n)_{n\geqslant 0}$ is positive definite, $\boldsymbol{u}$ is said to be a positive-definite functional.

Definition 1.3. Given a polynomial sequence $(P_n)_{n\geqslant 0}$, we say that $(P_n)_{n\geqslant 0}$ is a monic orthogonal polynomial sequence (MOPS in short) with respect to $\boldsymbol{u}$, and we denote it by $(P_n)_{n\geqslant 0} = \mathrm{mops}\,\boldsymbol{u}$ if and only if $P_n(x) = x^n +$ lower degree terms and $\langle \boldsymbol{u},P_mP_n\rangle = k_n\delta_{nm}$, $k_n\neq 0$, $n\geqslant 0$.

The next theorem will also be useful.

Theorem 1.4 (Favard's Theorem [7]). *Let $(P_n)_{n\geqslant 0}$ be a monic polynomial basis sequence. Then, $(P_n)_{n\geqslant 0}$ is an MOPS if and only if there exist two sequences of complex numbers $(d_n)_{n\geqslant 0}$ and $(g_n)_{n\geqslant 1}$, such that $g_n\neq 0$, $n\geqslant 1$ and*

$$xP_n = P_{n+1} + d_nP_n + g_nP_{n-1},\quad P_{-1}=0,\quad P_0=1,\ \ n\geqslant 0, \tag{1.1}$$

where $P_{-1}(x)\equiv 0$ and $P_0(x)\equiv 1$. Moreover, the functional $\boldsymbol{u}$ such that the polynomials $(P_n)_{n\geqslant 0}$ are orthogonal with respect to it is positive-definite if and only if $(d_n)_{n\geqslant 0}$ is a real sequence and $g_n>0$ for all $n\geqslant 1$.

In the following, we will use the notation:

Definition 1.5. Let $\pi\in\mathbb{P}$ and $a\in\mathbb{C}$, $a\neq 0$. We call the operator $H_a:\mathbb{P}\to\mathbb{P}$, $H_a\pi(x)=\pi(ax)$, a dilation of ratio $a\in\mathbb{C}\backslash\{0\}$.

This operator is linear on $\mathbb{P}$ and $H_a(\pi\rho)=H_a\pi\cdot H_a\rho$. Also notice that for any complex number $a\neq 0$, $H_a\cdot H_{a^{-1}}=I$, where I is the identity operator on $\mathbb{P}$, i.e., for all $a\neq 0$, H_a has an inverse operator. In the following we will omit any reference to q in the operators H_q and their inverse $H_{q^{-1}}$. So, $H := H_q$, $H^{-1} := H_{q^{-1}}$.

Next, we will introduce the so called q-derivative operator [11]. We will also suppose that $|q| \neq 1$ (although it is possible to weaken this condition).

Definition 1.6. Let $\pi \in \mathbb{P}$ and $q \in \mathbb{C}\backslash\{0\}$, $|q| \neq 1$. The q-derivative operator Θ, is the operator $\Theta : \mathbb{P} \to \mathbb{P}$,

$$\Theta\pi = \frac{H\pi - \pi}{Hx - x} = \frac{H\pi - \pi}{(q-1)x}.$$

The q^{-1}-derivative operator $\Theta^\star$, is the operator $\Theta^\star : \mathbb{P} \to \mathbb{P}$

$$\Theta^\star\pi = \frac{H^{-1}\pi - \pi}{H^{-1}x - x} = \frac{H^{-1}\pi - \pi}{(q^{-1}-1)x}.$$

Here, $\Theta\pi$ and $\Theta^\star\pi$ will denote the q-derivative and q^{-1}-derivative of π, respectively.

The above two operators Θ and $\Theta^\star$ are linear operators on $\mathbb{P}$, and

$$\Theta x^n = \frac{Hx^n - x^n}{(q-1)x} = \frac{(q^n - 1)x^n}{(q-1)x} = [n]x^{n-1}, \quad n > 0, \quad \Theta 1 = 0, \tag{1.2}$$

i.e., $\Theta\pi \in \mathbb{P}$. Here $[n]$, $n \in \mathbb{N}$, denotes the basic q-number n defined by

$$[n] = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}, \quad n > 0, \quad [0] = 0. \tag{1.3}$$

Also the q^{-1} numbers $[n]^\star$, defined by $[n]^\star = (q^{-n} - 1)/(q^{-1} - 1) = q^{1-n}[n]$ will be used.

Notice that $\Theta^\star$ is not the inverse of Θ. In fact, they are related by $H\Theta^\star = \Theta$, $\quad H^{-1}\Theta = \Theta^\star$. For the q-derivative the product rule $\Theta(\pi\rho) = \rho\Theta\pi + H\pi \cdot \Theta\rho = H\rho \cdot \Theta\pi + \pi\Theta\rho$, holds.

Definition 1.7. Let ω a derivable function at $x = 0$ such that $\forall a \in \operatorname{dom}\omega$, $aq \in \operatorname{dom}\omega$. Then, we will define the q-derivative of ω by

$$\Theta\omega(x) = \frac{H\omega(x) - \omega(x)}{Hx - x} = \frac{H\omega(x) - \omega(x)}{(q-1)x}, \quad x \neq 0, \quad \Theta\omega(0) = \omega'(0). \tag{1.4}$$

Definition 1.8. Let $\boldsymbol{u} \in \mathbb{P}^*$ and $\pi \in \mathbb{P}$. We define the action of a dilation H_a and the q-derivative Θ on $\mathbb{P}^*$ by $H_a : \mathbb{P}^* \to \mathbb{P}^*$, $\langle H_a\boldsymbol{u}, \pi\rangle = \langle \boldsymbol{u}, H_a\pi\rangle$, $\Theta : \mathbb{P}^* \to \mathbb{P}^*$, $\langle \Theta\boldsymbol{u}, \pi\rangle = -\langle \boldsymbol{u}, \Theta\pi\rangle$, respectively.

Definition 1.9. Let $\boldsymbol{u} \in \mathbb{P}^*$ and $\pi \in \mathbb{P}$. We define a polynomial modification of a functional $\boldsymbol{u}$, the functional $\pi\boldsymbol{u}$, $\langle \pi\boldsymbol{u}, \rho\rangle = \langle \boldsymbol{u}, \pi\rho\rangle$, $\forall\, \rho \in \mathbb{P}$.

Notice that we use the same notation for the operators on $\mathbb{P}$ and $\mathbb{P}^*$. Whenever it is not specified on which linear space an operator acts, it will be understood that it acts on the polynomial space $\mathbb{P}$.

Definition 1.10. Let $\boldsymbol{u} \in \mathbb{P}^*$ be a quasi-definite functional and $(P_n)_{n \geqslant 0} = \text{mops}(\boldsymbol{u})$. We say that $\boldsymbol{u}$ (resp. $(P_n)_{n \geqslant 0}$) is a q-classic functional (resp. a classical MOPS), if and only if the sequence $(\Theta P_{n+1})_{n \geqslant 0}$ is also orthogonal.

Notice that in the Hahn definition [11] q is a real parameter and here, in general, $q \in \mathbb{C}\backslash\{0\}$, $|q| \neq 1$. In the following $(Q_n)_{n \geqslant 0}$ will denote the sequence of monic q-derivatives of $(P_n)_{n \geqslant 0}$, i.e., $Q_n = (1/[n+1])P_{n+1}$, for all $n \geqslant 0$.

Theorem 1.11 (Medem et al. [17,18]). *Let $\boldsymbol{u} \in \mathbb{P}^*$ be a quasi-definite functional and* $(P_n)_{n \geqslant 0} = \text{mops}(\boldsymbol{u})$. *The following statements are equivalent*:
(a) *$\boldsymbol{u}$ and $(P_n)_{n \geqslant 0}$ are, respectively, a q-classical functional and a q-classical MOPS.*
(b) *There exist two polynomials ϕ and ψ, $\deg\phi \leqslant 2$, $\deg\psi = 1$, such that*

$$\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u}. \tag{1.5}$$

(c) *$(P_n)_{n \geqslant 0}$ satisfies the $q - \mathscr{SL}$ difference equation*

$$\phi\Theta\Theta^{\star}P_n + \psi\,\Theta^{\star}P_n = \hat{\lambda}_n P_n, \quad n \geqslant 0, \tag{1.6}$$

i.e., P_n are the eigenfunctions of the Sturm–Liouville operator $\phi\Theta\Theta^{\star} + \psi\,\Theta^{\star}$ associated with the eigenvalues $\hat{\lambda}_n$.

Moreover, if

$$\phi(x) = \hat{a}x^2 + \bar{a}x + \dot{a}, \qquad \psi(x) = \hat{b}x + \bar{b}, \quad \hat{b} \neq 0, \tag{1.7}$$

and $\boldsymbol{u}$ is quasi-definite then $[n]\hat{a} + \hat{b} \neq 0$ and the following equivalences hold:

$$[n]\hat{a} + \hat{b} \neq 0, \quad n \geqslant 0 \;\Leftrightarrow\; \hat{\lambda}_n \neq \hat{\lambda}_m, \quad \forall n, m \geqslant 1, n \neq m \;\Leftrightarrow\; \hat{\lambda}_n \neq 0, \quad \forall n \geqslant 1.$$

Theorem 1.12. *Let $\boldsymbol{u} \in \mathbb{P}^*$ be a quasi-definite functional, $(P_n)_{n \geqslant 0} = \text{mops}\,\boldsymbol{u}$ and $Q_n^{(k)} = (1/[n+1]_{(k)})\Theta^k P_{n+k}$, where $[n+1]_{(k)} \equiv [n+1][n+2]\ldots[n+k-1]$. The following statements are equivalent*: (a) *$(P_n)_{n \geqslant 0}$ is q-classical*, (b) *$(Q_n^{(k)})_{n \geqslant 0}$ is q-classical, $k \geqslant 1$.*

Moreover, if $\boldsymbol{u}$ satisfies the equation $\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u}$, $\deg\phi \leqslant 2$ and $\deg\psi = 1$, then $(Q_n^{(k)})$ is orthogonal with respect to $\boldsymbol{v}^{(k)} = H^{(k)}\phi \cdot \boldsymbol{u}$, $H^{(k)} = \prod_{i=1}^{k} H^{i-1}\phi$, and it satisfies

$$\Theta(\phi^{(k)}\boldsymbol{v}^{(k)}) = \psi^{(k)}\boldsymbol{v}^{(k)}, \qquad \deg\phi^{(k)} \leqslant 2 \quad \deg\psi^{(k)} = 1,$$

where $\phi^{(k)} = H^k\phi$ and $\psi^{(k)} = \psi + \Theta\sum_{i=0}^{k-1} H^i\phi$, and they are the polynomial solutions of the $q - \mathscr{SL}$ equation

$$\mathscr{SL}^{(k)}Q_n^{(k)} = \phi^{(k)}\Theta\Theta^{\star}Q_n^{(k)} + \psi^{(k)}\Theta^{\star}Q_n^{(k)} = \hat{\lambda}_n^{(k)}Q_n^{(k)}, \tag{1.8}$$

where the polynomials $\phi^{(k)}$ and $\psi^{(k)}$ and the eigenvalues $\hat{\lambda}_n^{(k)}$ are

$$\phi^{(k)} = q^{2k}\hat{a}x^2 + q^k\bar{a}x + \dot{a}, \qquad \psi^{(k)} = ([2k]\hat{a} + \hat{b})x + ([k]\bar{a} + \bar{b}),$$

$$\hat{\lambda}_n^{(k)} = [n]^{\star}([2k+n-1]\hat{a} + \hat{b}). \tag{1.9}$$

Furthermore, in [17,18] the following result was proved:

Theorem 1.13. *Let $\boldsymbol{u} \in \mathbb{P}^*$ be a quasi-definite functional, $(P_n)_{n \geqslant 0} = \operatorname{mops} \boldsymbol{u}$, ϕ, $\phi^\star$, $\psi \in \mathbb{P}$, such that $\phi^\star = q^{-1}\phi + (q^{-1}-1)x\psi$, $\deg \phi \leqslant 2$, $\deg \phi^\star \leqslant 2$ and $\deg \psi = 1$. Then, the following statements are equivalent*:

(a) *$\boldsymbol{u}$ and $(P_n)_{n \geqslant 0} = \operatorname{mops} \boldsymbol{u}$ are q-classical and $\Theta(\phi \boldsymbol{u}) = \psi \boldsymbol{u}$,*

(b) *$\boldsymbol{u}$ and $(P_n)_{n \geqslant 0} = \operatorname{mops} \boldsymbol{u}$ are q^{-1}-classical and $\Theta^\star(\phi^\star \boldsymbol{u}) = \psi \boldsymbol{u}$.*

(c) *There exist $\phi \in \mathbb{P}$, $\deg \phi \leqslant 2$ and three sequences of complex numbers a_n, b_n, c_n, $c_n \neq 0$, such that*

$$\phi \Theta P_n = a_n P_{n+1} + b_n P_n + c_n P_{n-1}, \quad n \geqslant 1; \tag{1.10}$$

(d) *there exist two sequences of complex numbers e_n, h_n, such that*

$$P_n = Q_n + e_n Q_{n-1} + h_n Q_{n-2}, \quad n \geqslant 2. \tag{1.11}$$

(e) *There exist $\phi \in \mathbb{P}$, $\deg \phi \leqslant 2$ and a sequence of complex numbers r_n, $r_n \neq 0$, $n \geqslant 1$ such that*

$$P_n \boldsymbol{u} = r_n \Theta^n (H^{(n)}\phi \cdot \boldsymbol{u}), \quad H^{(n)}\phi = \prod_{i=1}^{n} H^{i-1}\phi, \quad r_n = q^{\binom{n}{2}} \prod_{i=1}^{n} ([2n-i-1]\hat{a} + \hat{b})^{-1}, \quad n \geqslant 1. \tag{1.12}$$

2. The q-weight function ω

2.1. Definition and first properties

In this section we will consider the so-called weight functions for q-classical polynomials. The proof of the next proposition is straightforward (see, e.g., [12]).

Proposition 2.1. *Let ω be a complex function such that if $a \in \operatorname{dom} \omega$, $aq^{-1} \in \operatorname{dom} \omega$ and such that*

$$\Theta^\star(\phi\omega) = q\psi\,\omega \;\Leftrightarrow\; \phi\omega = qH(\phi^\star\omega), \quad \phi, \psi \in \mathbb{P}, \qquad \phi^\star = q^{-1}\phi + (q^{-1}-1)x\psi. \tag{2.1}$$

Then,

$$\phi \Theta \Theta^\star P_n + \psi\, \Theta^\star P_n = \hat{\lambda}_n P_n, \;\Leftrightarrow\; \Theta^\star(\phi\omega\Theta P_n) = q\hat{\lambda}_n \omega P_n, \quad n \geqslant 1. \tag{2.2}$$

The above proposition allows us to generalize to the q-case the classical procedure for obtaining almost all the main properties of the MOPS, in particular the q-Rodrigues formula from where expressions similar to (1.10) and (1.11) easily follow [2].

Eq. (2.1) is usually called the q-Pearson equation and its solution ω is known as the q-weight function. It allows to rewrite the Sturm–Liouville equation (1.6) in its self-adjoint form (2.2). Moreover, the weight function ω allows us to obtain the "standard" q-Rodrigues formula and also to justify the q-integral representation for the orthogonality relation. In such a case it is natural to give the following

Definition 2.2. Let $\boldsymbol{u} \in \mathbb{P}^*$ be a quasi-definite functional satisfying the distributional equation (1.5), where $\phi, \psi \in \mathbb{P}$, $\deg \phi \leqslant 2$, $\deg \psi = 1$ and $(P_n)_{n \geqslant 0} = \operatorname{mops} \boldsymbol{u}$. ω is said to be the q-weight function associated with $\boldsymbol{u}$ (resp. with $(P_n)_{n \geqslant 0}$) if ω satisfies Eq. (2.1) $\Theta^\star(\phi\omega) = q\psi\,\omega$.

The last definition allows us to rewrite the $q-\mathscr{SL}$ equation (1.8) in its self-adjoint form. In fact, straightforward calculations show that, if $\omega^{(k)}$ satisfies the q-Pearson equation

$$\Theta^\star(\phi^{(k)}\omega^{(k)}) = q\psi^{(k)}\omega^{(k)}, \tag{2.3}$$

where $\phi^{(k)}$ and $\psi^{(k)}$ are given in (1.9), then (1.8) can be rewritten in its self-adjoint form

$$\Theta^\star(\phi^{(k)}\omega^{(k)}\Theta Q_n^{(k)}) = q\hat{\lambda}_n^{(k)}\omega^{(k)}Q_n^{(k)}, \quad n \geqslant 1, \ \ k = 0, 1, \ldots, n. \tag{2.4}$$

Proposition 2.3. *Let* ω *be the solution of* (2.1) *and* $\omega^{(k)}$ *the solution of* (2.3). *Then,*

$$\omega^{(k)} = \phi^{(k-1)}\omega^{(k-1)} = \cdots = H^{(k)}\phi \cdot \omega, \quad \omega^{(0)} \equiv \omega. \tag{2.5}$$

Proof. We start from the q-Pearson equation (2.3) and rewrite it in the equivalent form $\phi^{(k)}\omega^{(k)} = qH[\phi^{(k)}]^\star H\omega^{(k)}$, where $[\phi^{(k)}]^\star = q^{-1}\phi^{(k)} + (q^{-1}-1)x\psi^{(k)} = \phi^\star$, for all $k \in \mathbb{N}$. Thus, by substituting $\omega^{(k)} = H^{(k)}\phi \cdot \omega$ in $\phi^{(k)}\omega^{(k)} = qH\phi^\star H\omega^{(k)}$, we find

$$\phi^{(k)}\omega^{(k)} = qH[\phi^{(k)}]^\star H\omega^{(k)} \ \Leftrightarrow \ H^k\phi(\phi H\phi\cdots H^{k-1}\phi\cdot\omega) = qH\phi^\star H\phi\cdots H^k\phi H\omega$$

$$\Leftrightarrow \ \phi\omega = qH\phi^\star H\omega \ \Leftrightarrow \ \Theta^\star(\phi\omega) = q\psi\,\omega,$$

from where the proposition follows. $\square$

Remark 2.4. Notice that the polynomials $(\phi^{(k)})^\star$ and $(\phi^\star)^{(k)}$ are very different. In fact, the first one together with $\psi^{(k)}$ are the polynomials which appear in the q^{-1}-distributional equation satisfied by the functional $\boldsymbol{v}^{(k)}$, i.e., the functional with respect to which the kth monic derivatives $Q_n^{(k)}$ are orthogonal, (see Proposition 1.4)

$$\Theta(\phi^{(k)}\boldsymbol{v}^{(k)}) = \psi^{(k)}\boldsymbol{v}^{(k)} \ \Leftrightarrow \ \Theta^\star(\phi^{(k)})^\star\boldsymbol{v}^{(k)} = \psi^{(k)}\boldsymbol{v}^{(k)}, \qquad (\phi^{(k)})^\star = \phi^\star, \quad \forall k \in \mathbb{N},$$

whereas the second one and $(\psi^\star)^{(k)}$ are the polynomial coefficients of the $q^{-1}-\mathscr{SL}$ equation

$$(\psi^\star)^{(k)}\Theta^\star\Theta Q_n^{\star(k)}(\psi^\star)^{(k)} = (\hat{\lambda}^\star)_n^{(k)}Q_n^{\star(k)},$$

that $Q_n^{\star(k)} = (1/[n+1]_{(k)}^\star)[\Theta^\star]^n P_{n+k}$ satisfy, or the q^{-1}-distributional equation is satisfied by the functional $\boldsymbol{v}^{\star(k)}$,

$$\Theta^\star[(\phi^\star)^{(k)}\boldsymbol{v}^{\star(k)}] = (\psi^\star)^{(k)}\boldsymbol{v}^{\star(k)}, \qquad (\phi^\star)^{(k)} = H^{-k}\phi^\star, \quad \forall k \in \mathbb{N},$$

i.e., the functional with respect to which the kth monic derivatives $Q_n^{\star(k)}$ are orthogonal.

2.2. Computation of the q-weight functions

In this section we will obtain the q-weight function associated with all q-classical functionals, i.e., the quasi-definite functionals corresponding to the MOPS in the widespread sense $\langle \boldsymbol{u}, P_n^2\rangle \neq 0$, for all $n \geqslant 0$. In fact, Theorems 2.6 and 2.8 will give, in a very natural way, the key for the classification of all q-classical orthogonal polynomials.

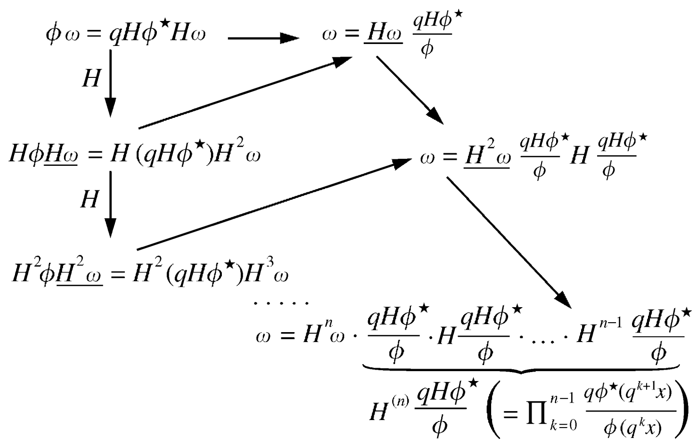

Fig. 1. Recurrent schema using the q-dilation.

In the following, we consider the case when $|q| < 1$ ($|q^{-1}| > 1$). Also we will use the standard notation $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$, $(a;q)_0 \equiv 1$ for the q-analogue of the Pochhammer symbol, and $(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$, for the absolutely convergent infinite product when $|q| < 1$. First of all, we will rewrite the q-Pearson equation (2.1)

$$\Theta^\star(\phi\omega) = q\psi\,\omega \;\Leftrightarrow\; \phi\omega = q(H\phi^\star)(H\omega) \;\Leftrightarrow\; \phi^\star\omega = q^{-1}(H^{-1}\phi)(H^{-1}\omega) \tag{2.6}$$

and solve the resulting equation by the recurrent procedure shown in Fig. 1.

When ω is continuous at 0 and $\omega(0) \neq 0$, taking the limit $n \to \infty$, we find, since $\lim_{n\to\infty} H^n\omega = \lim_{n\to\infty}\omega(q^n x) = \omega(0)$,

$$\omega = \omega(0)\,H^{(\infty)}\frac{qH\phi^\star}{\phi} = \omega(0)\lim_{n\to\infty} H^{(n)}\frac{qH\phi^\star}{\phi} = \omega(0)\prod_{n=0}^{\infty}\frac{qH\phi^\star(q^{k+1}x)}{\phi(q^k x)}. \tag{2.7}$$

The next step leads us to an explicit expression for the product $H^{(\infty)}(qH\phi^\star/\phi)$. For doing that we need a lemma which is interesting in its own right.

Lemma 2.5. *If π is an nth degree polynomial with an independent term $\pi(0) = 1$, and zeros $a_i \in \mathbb{C}\backslash\{0\}$, $i = 1,2,\ldots,n$, then*

$$H^{(\infty)}\pi = (a_1^{-1}x;q)_\infty(a_2^{-1}x;q)_\infty\cdots(a_n^{-1}x;q)_\infty := (a_1^{-1}x, a_2^{-1}x,\ldots,a_n^{-1}x;q)_\infty$$

is an entire function of x with zeros at $a_i q^{-k}$, $i = 1,2,\ldots,n$ and $k \geqslant 0$. Furthermore, if π/ρ is a rational function such that $\pi(0) = \rho(0) \neq 0$ and with nonzero poles and zeros, then,

$$H^{(\infty)}\frac{\pi}{\rho} = \frac{(a_1^{-1}x;q)_\infty(a_2^{-1}x;q)_\infty\cdots(a_n^{-1}x;q)_\infty}{(b_1^{-1}x;q)_\infty(b_2^{-1}x;q)_\infty\cdots(b_m^{-1}x;q)_\infty} = \frac{(a_1^{-1}x, a_2^{-1}x,\ldots,a_n^{-1}x;q)_\infty}{(b_1^{-1}x, b_2^{-1}x,\ldots,b_m^{-1}x;q)_\infty}$$

is a meromorphic function with zeros at $a_i q^{-k}$, $i=1,2,\ldots,n$ *and* $k \geqslant 0$ *and poles at* $b_j q^{-l}$, $j=1,2,\ldots,m$ *and* $l \geqslant 0$, *where* $a_i \in \mathbb{C}$, $i=1,2,\ldots,n$ *and* $b_k \in \mathbb{C}$, $k=1,2,\ldots,m$, *are the zeros and poles of* π/ρ, *respectively.*

Proof. The proof is based on the fact that, if π is a polynomial of degree n such that $\pi(0)=1$, then

$$\pi = A(x-a_1)(x-a_2)\cdots(x-a_n) = \underbrace{(-1)^n A a_1 a_2 \cdots a_n}_{\pi(0)=1}(1-a_1^{-1}x)(1-a_2^{-1}x)\cdots(1-a_n^{-1}x).$$

Thus, $H^{(k)}\pi = (a_1^{-1}x, a_2^{-1}x, \ldots, a_n^{-1}x; q)_k$ and so, $H^{(\infty)}\pi = (a_1^{-1}x, a_2^{-1}x, \ldots, a_n^{-1}x; q)_\infty$. This function is an entire function according to the Weierstrass Theorem (see, e.g., [1, Section 4.3]). The proof of the second statement is analogous and the function $H^{(\infty)}(\pi/\rho)$ is meromorphic as a quotient of two entire functions (see, e.g., [1, Section 4.3]). □

Now, if $\phi(0) \neq 0$, the above lemma yields the following well-known result [11].

Theorem 2.6. *Let* $(P_n)_{n\geqslant 0} = \mathrm{mops}\,\boldsymbol{u}$ *satisfy the q-Sturm–Liouville equation* (1.6). *If we denote by* a_1 *and* a_2 *the zeros of* ϕ *and by* $a_1^\star$ *and* $a_2^\star$ *the zeros of* $\phi^\star$ (*see Proposition* 1.5), *and all of them are different from* 0, *then the following expressions for the q-weight function* ω *hold*:

ϕ	$\phi^\star$	q-Weight function $\omega(x)$
$\hat{a}(x-a_1)(x-a_2)$, $\hat{a}a_1a_2 \neq 0$	$\hat{a}^\star(x-a_1^\star)(x-a_2^\star)$, $\hat{a}^\star a_1^\star a_2^\star \neq 0$	$\omega(x) = \dfrac{(a_1^{\star-1}qx,\, a_2^{\star-1}qx; q)_\infty}{(a_1^{-1}x,\, a_2^{-1}x; q)_\infty}$
$\hat{a}(x-a_1)(x-a_2)$, $\hat{a}a_1a_2 \neq 0$	$\bar{a}^\star(x-a_1^\star)$, $\bar{a}^\star a_1^\star \neq 0$	$\omega(x) = \dfrac{(a_1^{\star-1}qx; q)_\infty}{(a_1^{-1}x,\, a_2^{-1}x; q)_\infty}$
$\hat{a}(x-a_1)(x-a_2)$, $\hat{a}a_1a_2 \neq 0$	$\dot{a}^\star \neq 0$	$\omega(x) = \dfrac{1}{(a_1^{-1}x,\, a_2^{-1}x; q)_\infty}$
$\bar{a}(x-a_1)$, $\bar{a}a_1 \neq 0$	$\hat{a}(x-a_1)(x-a_2)$, $\hat{a}a_1a_2 \neq 0$	$\omega(x) = \dfrac{(a_1^{\star-1}qx,\, a_2^{\star-1}qx; q)_\infty}{(a_1^{-1}x; q)_\infty}$
$\dot{a} \neq 0$	$\hat{a}(x-a_1)(x-a_2)$, $\hat{a}a_1a_2 \neq 0$	$\omega(x) = (a_1^{\star-1}qx,\, a_2^{\star-1}qx; q)_\infty$

Proof. Since $\phi(x) = \hat{a}(x-a_1)(x-a_2)$ and $\phi^\star = q^{-1}\phi + (q^{-1}-1)x\psi = \hat{a}^\star(x-a_1^\star)(x-a_2^\star)$, we get $(qH\phi^\star)(0) = q\phi^\star(0) = \phi(0)$. Thus the polynomials $qH\phi^\star$ and ϕ have the same independent term. From

$$\phi(x) = \hat{a}x^2 + \bar{a}x + \dot{a}, \qquad \phi^\star(x) = \hat{a}^\star x^2 + \bar{a}^\star x + \dot{a}^\star,$$

we have $\hat{a}^\star = q^{-1}\hat{a} + (q^{-1}-1)\hat{b}$, $\bar{a}^\star = q^{-1}\bar{a} + (q^{-1}-1)\bar{b}$ and $\dot{a}^\star = q^{-1}\dot{a}$, where, $\hat{b}, \bar{b}$ are given in Eq. (1.7). Thus,

$$\deg\phi < 2 \;\Rightarrow\; \hat{a}=0 \;\Rightarrow\; \hat{a}^\star \neq 0 \;\Rightarrow\; \deg\phi^\star = 2,$$

$$\deg\phi = 2 \;\Rightarrow\; \hat{a}\neq 0 \begin{cases} \hat{b} \neq -\frac{\hat{a}}{1-q} \;\Rightarrow\; \hat{a}^\star \neq 0 \;\Rightarrow\; \deg\phi^\star = 2, \\ \hat{b} = -\frac{\hat{a}}{1-q} \;\Rightarrow\; \hat{a}^\star = 0 \begin{cases} \bar{b} \neq -\frac{\bar{a}}{1-q} \;\Rightarrow\; \deg\phi^\star = 1, \\ \bar{b} = -\frac{\bar{a}}{1-q} \;\Rightarrow\; \deg\phi^\star = 1. \end{cases} \end{cases}$$

In all the cases the above lemma leads us to the desired result. Notice also that all the obtained functions are meromorphic and so, they are continuous and do not vanish at $x=0$, so we can suppose without any loss of generality that $\omega(0)=1$. □

When $\phi(0)=0$, it is easy to see that $\phi^\star(0)=0$. This case requires a more careful study. In the following we should keep in mind that for the quasi-definiteness of $\boldsymbol{u}$, $\phi\neq 0$ and ϕ and ψ should be coprime polynomials (see [18]).

Proposition 2.7. *Let $\boldsymbol{u}$ be a q-classical functional satisfying the distributional equation* (1.5) *with $\phi=\hat{a}x^2+\bar{a}x$, $|\hat{a}|+|\bar{a}|>0$, and $\psi=\hat{b}x+\bar{b}$, $\hat{b}\neq 0$. Then the following cases, compatible with the quasi-definiteness of $\boldsymbol{u}$, appear*:

(a) *If $\phi=\hat{a}x^2$, $\hat{a}\neq 0$, then* $\deg\phi^\star=2$ *with simple zeros, or* $\deg\phi^\star=1$.

(b) *If $\phi=\hat{a}x^2+\bar{a}x$, $\hat{a}\bar{a}\neq 0$, then* $\deg\phi^\star=2$ *or* $\deg\phi^\star=1$.

(c) *If $\phi=\bar{a}x$, $\bar{a}\neq 0$, then* $\deg\phi^\star=2$.

Proof. (a) Since $\phi=\hat{a}x^2$, then $\psi=\hat{b}x+\bar{b}$, with $\bar{b}\neq 0$, otherwise ψ divides ϕ. Therefore, the coefficient on x of the polynomial $\phi^\star=(q^{-1}\hat{a}+(q^{-1}-1)\hat{b})x^2+(q^{-1}-1)\bar{b}x$ is different from zero. If $\hat{b}\neq -\hat{a}/(1-q)$ then $\hat{a}^\star\neq 0$ and $\deg\phi^\star=2$ and $\phi^\star$ has two simple zeros and one of them is located at the origin. If $\hat{b}=-\hat{a}/(1-q)$ then $\deg\phi^\star=1$. The other two cases are proved analogously. □

The next step is to find the q-weight functions for all possible cases according to the above proposition (remember that $\phi(0)=0=\phi^\star(0)$). There are two large classes. Class I corresponds to the case when ϕ and $\phi^\star$ have a nonvanishing coefficient on x and II when they have a vanishing coefficient on x.

(I) We start with the case when ϕ and $\phi^\star$ have a nonvanishing coefficient on x. In this case there are three different possibilities (subclasses):

(a) $\phi(x)=\hat{a}x(x-a_1)$, $\hat{a}a_1\neq 0$ and $\phi^\star(x)=\hat{a}^\star x(x-a_1^\star)$, $\hat{a}^\star a_1^\star\neq 0$,

(b) $\phi(x)=\hat{a}x(x-a_1)$, $\hat{a}a_1\neq 0$ and $\phi^\star(x)=\bar{a}^\star x$, $\bar{a}^\star\neq 0$,

(c) $\phi(x)=\bar{a}x$, $\bar{a}\neq 0$ and $\phi^\star(x)=\hat{a}^\star x(x-a_1^\star)$, $\hat{a}^\star a_1^\star\neq 0$.

To find the corresponding q-weight functions we will rewrite the quotient $qH\phi^\star/\phi = xq\,(H\phi^\star)_0/x\phi_0$, where $\phi=x\phi_0$ and $H\phi^\star=x(H\phi^\star)_0$. In general, $\phi_0(0)\neq(H\phi^\star)_0(0)$; so, in order to apply a method, similar to the one used to prove Theorem 2.1, we will assume that ω can be rewritten in the form $\omega=|x|^\alpha\omega_0$, $\alpha\in\mathbb{C}\backslash\{0\}$, where α is a free parameter to be found. Straightforward calculations show that if ω satisfies a q-Pearson equation (2.6) then $\phi_0\omega_0=aq(H\omega_0)(H\phi^\star)_0$ holds, where $a=q^\alpha$. So,

$$\omega_0=(H\omega_0)\frac{aq(H\phi^\star)_0}{\phi_0}, \quad a=q^\alpha, \quad \text{or} \;\; \alpha=\mathrm{Log}_q(a),$$

where Log_q denotes the principal logarithm on the basis q [1], $|q| < 1$. In the following, we will use the notation $\bar{a} = -\hat{a}a_1$ and $\bar{a}^\star = -\hat{a}^\star a_1^\star$. Notice that, with this notation, $\phi = \hat{a}x(x-a_1) = \hat{a}x^2 + \bar{a}x$ and $\phi^\star = \hat{a}^\star x(x-a_1^\star) = \hat{a}^\star x^2 + \bar{a}^\star x$.

(a) In this case,

$$\frac{aq(H\phi^\star)_0}{\phi_0} = \frac{aq^2\hat{a}^\star a_1^\star(a_1^{\star-1}qx-1)}{\hat{a}a_1(a_1^{-1}x-1)} = \frac{aq^2\bar{a}^\star(1-a_1^{\star-1}qx)}{\bar{a}(1-a_1^{-1}x)}.$$

If we choose a such that $aq(H\phi^\star)_0(0) = \phi_0(0)$, i.e., $aq^2\bar{a}^\star = \bar{a}$, or, equivalently, $\alpha = \mathrm{Log}_q(a) = -2 + \mathrm{Log}_q \bar{a}/\bar{a}^\star$, we can apply Lemma 2.5 to get, $\omega_0 = \omega_0(0)(a_1^{\star-1}qx;q)_\infty/(a_1^{-1}x;q)_\infty$, which leads, without loss of generality, to the following weight function (here we assume that ω_0 is continuous and $\omega_0(0) \neq 0$):

$$\omega(x) = |x|^\alpha \frac{(a_1^{\star-1}qx;q)_\infty}{(a_1^{-1}x;q)_\infty}, \qquad \alpha = \mathrm{Log}_q(a) = -2 + \mathrm{Log}_q \frac{\bar{a}}{\bar{a}^\star}. \tag{2.8}$$

(b) In this case,

$$\frac{aq(H\phi^\star)_0}{\phi_0} = \frac{aq^2\bar{a}^\star}{\bar{a}(1-a_1^{-1}x)} \;\Rightarrow\; \omega_0 = \omega_0(0)\frac{1}{(a_1^{-1}x;q)_\infty}, \qquad \alpha = \mathrm{Log}_q(a) = -2 + \mathrm{Log}_q\left(-\frac{\bar{a}}{\bar{a}^\star}\right),$$

so,

$$\omega(x) = \frac{|x|^\alpha}{(a_1^{-1}x;q)_\infty}, \qquad \alpha = \mathrm{Log}_q(a) = -2 + \mathrm{Log}_q\left(-\frac{\bar{a}}{\bar{a}^\star}\right).$$

Finally, in the last case (c), we obtain

$$\omega(x) = |x|^\alpha (a_1^{\star-1}qx;q)_\infty, \qquad \alpha = \mathrm{Log}_q(a) = -2 + \mathrm{Log}_q\left(-\frac{\bar{a}}{\bar{a}^\star}\right).$$

(II) Let us consider the other case, i.e., when ϕ and $\phi^\star$ have a vanishing coefficient of x. In this case there are two possibilities:

(i) $\deg\phi \neq \deg\phi^\star$ which is divided in two subcases (a) $\phi = \hat{a}x^2$, $\phi^\star = \bar{a}^\star x$, and (b) $\phi = \bar{a}x$, $\phi^\star = \hat{a}^\star x^2$, and

(ii) $\deg\phi = \deg\phi^\star$, which is also divided into subcases (a) $\phi = \hat{a}x^2$, $\phi^\star = \hat{a}^\star x(x-a_1^\star)$, $a_1^\star \neq 0$, and (b) $\phi = \hat{a}x(x-a_1)$, $\phi^\star = \hat{a}^\star x^2$, $a_1 \neq 0$.

In both situations, the method used in case (I) of nonvanishing coefficients cannot be used.

(i) In order to solve the problem for case (II) (i) we will generalize an idea by Häcker [12]. Let us define the function $h^{(\beta)} : [0,\infty) \to \mathbb{R}$ defined by

$$h^{(\beta)}(x) = \sqrt{x^{\log_q x^\beta - \beta}}, \quad \beta \neq 0,$$

which has the following property $Hh^{(\beta)} = x^\beta h^{(\beta)}$, or, equivalently, $h^{(\beta)}(qx) = x^\beta h^{(\beta)}(x)$, for all $x \geqslant 0$.

If we now define the function $\omega = x^\alpha h^{(1)}$, then, for case (II) (i) (a) we get

$$H\omega = H(x^\alpha h^{(1)}) = q^\alpha x^\alpha x h^{(1)} = q^\alpha x\omega \;\Rightarrow\; xH\omega = q^\alpha x^2\omega,$$

then comparing this resulting equation with the q-Pearson equation (2.6) for this choice of ϕ and $\phi^\star$, $\hat{a}x^2\omega = q\bar{a}^\star xH\omega$, we deduce that the function

$$\omega(x) = |x|^\alpha \sqrt{x^{\log_q x - 1}}, \qquad \alpha = -2 + \mathrm{Log}_q \frac{\hat{a}}{\bar{a}^\star}, \quad x \geqslant 0 \tag{2.9}$$

is a solution of the q-Pearson equation and hence, the corresponding q-weight function.

For case II(i) (b) we have, in an analogous way, a similar solution but involving the function $h^{(-1)}$:

$$\omega(x)=|x|^{\alpha}\sqrt{x^{\log_q \frac{1}{x}+1}}, \qquad \alpha=-3+\mathrm{Log}_q\frac{\bar{a}}{\hat{a}^{\star}}, \quad x\geqslant 0. \tag{2.10}$$

(ii) In this case the method developed for the above cases does not work. In fact, if we try to use the method for case (I), after some straightforward calculations we arrive to a divergent infinite product. For this reason we will solve the q-Pearson equation using the equivalent equation (2.6) in q^{-1} dilation $q^{-1}H^{-1}\phi H^{-1}\omega=\phi^{\star}\omega$ (2.6), i.e., using a similar schema to the one given in Fig. 1 but when the recurrence is solved in the "opposite" direction to obtain the expression $\omega=(H^{-n}\omega)(H^{(-n)}q^{-1}(H^{-1}\phi)/\phi^{\star})$, which leads to the solution, by taking the limit $n\to\infty$, if there exists the value $H^{-\infty}\omega=\omega(\infty)$. In such a way, we have for case (II) (ii) (a) the expression

$$\omega=|x|^{\alpha}\omega_0, \quad \alpha\geqslant 0, \qquad \phi^{\star}=x\phi_0^{\star}=x(\hat{a}^{\star}x+\bar{a}^{\star}), \qquad H^{-1}\phi=x(H^{-1}\phi)_0=x(q^{-2}\hat{a}x),$$

hence, the q^{-1}-Pearson equation becomes

$$x\phi_0^{\star}\cdot|x|^{\alpha}\omega_0=q^{-1}\cdot x(H^{-1}\phi)_0\cdot H^{-1}(|x|^{\alpha}\omega_0) \;\Rightarrow\; \phi_0^{\star}\omega_0=q^{-1}(H^{-1}\phi)_0\,q^{-\alpha}H^{-1}\omega_0$$

and its solution is

$$\omega_0(0)=H^{-n}\omega_0H^{(-n)}\frac{q^{-\alpha}q^{-1}(H^{-1}\phi)_0}{\phi_0^{\star}} \overset{a:=q^{\alpha}}{=} H^{-n}\omega_0\,H^{(-n)}\frac{a^{-1}q^{-3}\hat{a}x}{\hat{a}^{\star}x+\bar{a}^{\star}}.$$

Now, choosing the value α in such a way that $a^{-1}q^{-3}\hat{a}=\hat{a}^{\star}$, i.e., $\alpha=-3+\mathrm{Log}_q\,\hat{a}/\hat{a}^{\star}$ we find,

$$\omega_0(0)=H^{-n}\omega_0\,H^{(-n)}\frac{\hat{a}^{\star}x}{\hat{a}^{\star}x+\bar{a}^{\star}}=H^{-n}\omega_0H^{(-n)}\left(1-\frac{\bar{a}^{\star}}{\hat{a}^{\star}x+\bar{a}^{\star}}\right)$$

$$=H^{-n}\omega_0\prod_{i=0}^{n}\left(1-\frac{\bar{a}^{\star}q^{i}}{\hat{a}^{\star}x+\bar{a}^{\star}q^{i}}\right).$$

Obviously, the above product is uniformly convergent in any compact subset of the complex plane that does not contain the points $\{a_1^{\star}q^n, n\geqslant 0\}\cup\{0\}$, where $a_1^{\star}=-\bar{a}^{\star}/\hat{a}^{\star}$ is the nonvanishing zero of $\phi^{\star}$ (in $x=0$ the product diverges to zero). Furthermore, this product converges at ∞, so $\omega(\infty)=c\neq 0$, and thus

$$\omega(x)=|x|^{\alpha}\omega_0=c|x|^{\alpha}\prod_{n=0}^{\infty}\left(1-\frac{\bar{a}^{\star}q^{n}}{\hat{a}^{\star}x+\bar{a}^{\star}q^{n}}\right)=c|x|^{\alpha}H^{(-\infty)}\frac{\hat{a}^{\star}x}{\hat{a}^{\star}x+\bar{a}^{\star}}=c|x|^{\alpha}\frac{1}{(-\bar{a}^{\star}/\hat{a}^{\star}x;q)_{\infty}},$$

where $\alpha=\mathrm{Log}_q\,\hat{a}q^{-3}/\hat{a}^{\star}$, which leads, without loss of generality, to the following expression for the q-weight function:

$$\omega(x)=|x|^{\alpha}\frac{1}{(a_1^{\star}/x;q)_{\infty}}=|x|^{\alpha}e_q(a_1^{\star}/x), \qquad a_1^{\star}=-\frac{\bar{a}^{\star}}{\hat{a}^{\star}x}, \qquad \alpha=-3+\mathrm{Log}_q\frac{\hat{a}}{\hat{a}^{\star}}, \tag{2.11}$$

where e_q denotes the q-exponential function [10].

A similar situation appears in case (II) (ii) (b) subcase. In this case,

$$\omega_0(x)=H^{-n}\omega_0H^{(-n)}\frac{q^{-\alpha}q^{-1}(H^{-1}\phi)_0}{\phi_0^{\star}} \overset{a:=q^{\alpha}}{=} H^{-n}\omega_0H^{(-n)}\frac{a^{-1}q^{-1}q^{-1}\hat{a}(q^{-1}x-a_1)}{\hat{a}^{\star}x}.$$

If we now choose $a^{-1}q^{-3}\hat{a}=\hat{a}^{\star}$, we find

$$\omega_0(x)=H^{-n}\omega_0H^{(-n)}\frac{\hat{a}^{\star}x-a^{-1}q^{-2}\hat{a}a_1}{\hat{a}^{\star}x}=H^{-n}\omega_0H^{(-n)}\left(1-\frac{\hat{a}^{\star}qa_1}{\hat{a}^{\star}x}\right)$$

$$=H^{-n}\omega_0H^{(-n)}\left(1-\frac{a_1q}{x}\right),$$

which is an absolutely and uniformly convergent product in $\mathbb{C}\backslash\{0\}$. Finally, since $\omega(\infty)=c\neq 0$, and, without loss of generality, we find

$$\omega(x)=|x|^{\alpha}(a_1q/x;q)_\infty, \qquad \alpha=-3-\operatorname{Log}_q\frac{\hat{a}}{\hat{a}^{\star}},$$

where a_1 is the nonvanishing zero of ϕ. We can then summarize all the above calculations in the following theorem:

Theorem 2.8. *Let* $(P_n)_{n\geqslant 0}=\text{mops}\,\boldsymbol{u}$ *satisfying the q-Sturm–Liouville equation* (1.6). *If we denote by* a_1 *and* a_2 *the zeros of* ϕ *and by* $a_1^{\star}$ *and* $a_2^{\star}$ *the zeros of* $\phi^{\star}$ (*see Proposition* 1.4), *and one of them is equal to* 0, *then the following expressions for the q-weight functions* ω *hold*:

Case	ϕ	$\phi^{\star}$	$\omega(x)$	
II(ii)a	$\hat{a}x^2, \hat{a}\neq 0$	$\hat{a}^{\star}x(x-a_1^{\star}), \hat{a}^{\star}a_1^{\star}\neq 0$	$\lvert x\rvert^{\alpha}\dfrac{1}{(a_1^{\star}/x;q)_\infty},$	$\alpha=\operatorname{Log}_q\dfrac{\hat{a}q^{-3}}{\hat{a}^{\star}}$
II(i)a	$\hat{a}x^2, \hat{a}\neq 0$	$\bar{a}^{\star}x,\ \bar{a}^{\star}\neq 0$	$\lvert x\rvert^{\alpha}\sqrt{x^{\log_q x-1}},$	$\alpha=\operatorname{Log}_q\dfrac{\hat{a}q^{-2}}{\bar{a}^{\star}}$
I(a)	$\hat{a}x(x-a_1)$, $\hat{a}a_1\neq 0$	$\hat{a}^{\star}x(x-\hat{a}_1^{\star}), \hat{a}^{\star}\hat{a}_1^{\star}\neq 0$	$\lvert x\rvert^{\alpha}\dfrac{(a_1^{\star-1}qx;q)_\infty}{(a_1^{-1}x;q)_\infty},$	$\alpha=\operatorname{Log}_q\dfrac{\bar{a}q^{-2}}{\bar{a}^{\star}}$
I(b)	$\hat{a}x(x-a_1)$, $\hat{a}a_1\neq 0$	$\bar{a}^{\star}x,\ \bar{a}^{\star}\neq 0$	$\lvert x\rvert^{\alpha}\dfrac{1}{(a_1^{-1}x;q)_\infty},$	$\alpha=\operatorname{Log}_q\dfrac{-\bar{a}q^{-2}}{\bar{a}^{\star}}$
II(ii)b	$\hat{a}x(x-a_1)$, $\hat{a}a_1\neq 0$	$\hat{a}^{\star}x^2,\ \hat{a}^{\star}\neq 0$	$\lvert x\rvert^{\alpha}(a_1q/x;q)_\infty,$	$\alpha=-\operatorname{Log}_q\dfrac{\bar{a}q^3}{\hat{a}^{\star}}$
I(c)	$\bar{a}x,\ \bar{a}\neq 0$	$\hat{a}^{\star}x(x-\hat{a}_1^{\star}), \hat{a}^{\star}a_1^{\star}\neq 0$	$\lvert x\rvert^{\alpha}(a_1^{\star-1}qx;q)_\infty,$	$\alpha=\operatorname{Log}_q\dfrac{-\bar{a}q^{-2}}{\bar{a}^{\star}}$
II(i)b	$\bar{a}x,\ \bar{a}\neq 0$	$\hat{a}^{\star}x^2,\ \hat{a}^{\star}\neq 0$	$\lvert x\rvert^{\alpha}\sqrt{x^{\log_q\frac{1}{x}+1}},$	$\alpha=\operatorname{Log}_q\dfrac{\bar{a}q^{-3}}{\hat{a}^{\star}}$

3. Applications

In this section we will consider some applications of the above theorems. In fact, we will show how the q-weight functions can be used to give an integral representation for the orthogonality. Another interesting application is the already mentioned classification of all *orthogonal* families of q-classical polynomials on the exponential lattice, i.e., the q-Hahn tableau (in [23] the orthogonality was not considered. A partial study was done in [8]). In fact, Theorems 2.6 and 2.8 give a natural classification of the q-classical orthogonal polynomials. Also by using the q-weights one can obtain an explicit formula of the polynomials satisfying a Rodrigues-type formula in terms of the polynomial coefficients of ϕ and $\phi^\star$. Thus the hypergeometric representation easily follows. This was done independently in [23,6] (see also [5,20]) in the framework of the difference equations of hypergeometric type on the nonuniform lattices. Here we will show how all the q-classical families can be obtained by certain limit processes from the most general case of $\emptyset$-Jacobi/Jacobi family. Finally, we will compare the Nikiforov–Uvarov and the q-Askey tableaus with our q-classical tableau and we will complete the q-Askey one with new families of orthogonal polynomials.

3.1. The q-integral representation for the orthogonality

In this section we will show how the q-weight functions and the $q-\mathscr{SL}$ equation lead to a q-integral representation for the orthogonality. The technique used here is standard in the theory of orthogonal polynomials (see, e.g., [7,13,20]). First of all, we introduce the q-integral of Jackson [10,25]. This integral is a Riemann sum on an infinite partition $\{aq^n, n \geqslant 0\}$ of the interval $[0,a]$

$$\int_0^a f(x)\,\mathrm{d}_q x = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n \quad \text{and} \quad \int_a^b f(x)\,\mathrm{d}_q x = \int_0^b f(x)\,\mathrm{d}_q x - \int_0^a f(x)\,\mathrm{d}_q x.$$

Furthermore, the q-analogue of the Barrow rule is true (here we assume that $\Theta F(x)$ is continuous at $x=0$): $\int_a^b \Theta F(x)\,\mathrm{d}_q x = F(b)-F(a)$, as well as the rules of integration by parts

$$\int_a^b f(x)\Theta g(x)\,\mathrm{d}_q x = H^{-1}f(x)\cdot g(x)\Big|_a^b - q\int_a^b g(x)\Theta^\star f(x)\,\mathrm{d}_q x,$$

$$\int_a^b f(x)\Theta g(x)\,\mathrm{d}_q x = fg\Big|_a^b - \int_a^b Hg(x)\Theta f(x)\,\mathrm{d}_q x.$$

Obviously, in the above expressions it is assumed that the function f is defined in the corresponding partition set. This Jackson q-integral can be easily generalized to unbounded intervals and unbounded functions in a similar way as the Riemann integral [10,25]. Furthermore, the Riemann–Stieltjes discrete integrals related to the q-classical polynomials can be represented as q-integrals (see, e.g., [17,19]).

Proposition 3.1. *Let ω be a continuous function in $x=0$ satisfying the q-Pearson equation $\Theta^\star(\phi\omega) = q\psi\,\omega$, which is equivalent to the distributional equation $\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u}$. Let a,b be complex numbers such that the boundary condition $\phi^\star\omega|_a^b = 0$, or equivalently, $H^{-1}\phi\omega|_a^b = 0$ $(\phi w = qH(\phi^\star w))$*

holds. Then

$$\int_a^b P_n(x)P_m(x)\omega(x)\,\mathrm{d}_qx = 0, \quad \forall n \neq m, \qquad (P_n)_{n\geqslant 0} = \mathrm{mops}\,\boldsymbol{u}.$$

Proof. The proof is straightforward. We start from the self-adjoint form of the $q-\mathscr{SL}$ equations for the polynomial P_n and P_m, respectively:

$$\Theta[H^{-1}(\phi\omega)\Theta^\star P_n] = \hat{\lambda}_n\omega P_n, \qquad \Theta[H^{-1}(\phi\omega)\Theta^\star P_m] = \hat{\lambda}_m\omega P_m.$$

If we multiply the first one by P_m, the second one by P_n, take the q-integral over (a,b), subtract and use the integration by part rules we find

$$\begin{aligned}(\hat{\lambda}_n - \hat{\lambda}_m)\int_a^b \omega P_n P_m\,\mathrm{d}_qx &= \int_a^b (\omega\hat{\lambda}_n P_n)P_m\,\mathrm{d}_qx - \int_a^b (\omega\hat{\lambda}_m P_m)P_n\,\mathrm{d}_qx \\ &= \int_a^b \Theta(H^{-1}(\phi\omega)\Theta^\star P_n)P_m\,\mathrm{d}_qx - \int_a^b \Theta(H^{-1}(\phi\omega)\Theta^\star P_n)P_n\,\mathrm{d}_qx \\ &= H^{-1}(\phi\omega)W_q^\star[P_m,P_n]|_a^b \\ &\quad + \int_a^b [H(H^{-1}(\phi\omega)\Theta^\star P_m)\Theta P_n - H(H^{-1}(\phi\omega)\Theta^\star P_n)\Theta P_m]\,\mathrm{d}_qx,\end{aligned}$$

where $W_q^\star[P_m,P_n] = P_m\Theta^\star P_n - P_n\Theta^\star P_m$ is the q-Wronskian. The first term in the last equation vanishes as a result of the boundary conditions. The second also vanishes since $H(H^{-1}(\phi\omega)\Theta^\star P_m)\Theta P_n = \phi\omega\Theta P_m\Theta P_n$. The result follows from the fact that for all $n\neq m$, $\hat{\lambda}_n \neq \hat{\lambda}_m$. □

Remark 3.2. Notice that the choice of the integration interval (a,b) is conditioned to guarantee that $\int_a^b P_n^2\omega\,\mathrm{d}_qx \neq 0$, $n\geqslant 0$, for which, it is enough that ω be a continuous function and does not vanish inside the interval of integration. This has a difficulty since, even in the simplest cases, i.e., $\emptyset$-families, ω has infinite zeros $a_i^\star q^{-n}$, $n\geqslant 1$, and infinite poles, a_iq^{-n}, $n\geqslant 0$. Notice also that natural values for (a,b) are the zeros of $\phi^\star$ or the zeros of $\phi(q^{-1}x)$.

The study of the positive definite case, i.e., the case when $\int_a^b P_n^2\omega\,\mathrm{d}_qx > 0$ for all $n\geqslant 0$ has a special interest and it will be considered in a forthcoming paper.

3.2. Classification of the q-classical polynomials

Since Eq. (1.5) (and so the Sturm–Liouville equation (2.2)) gives the basic information about the q-classical functional (and then about the corresponding MOPS), it is natural to use them in order to classify the q-classical polynomials. Moreover, all this information is condensed in the polynomials ϕ and $\phi^\star$ instead of ϕ and ψ (and more exactly in their zeros) as it is shown in Theorems 2.6 and 2.8. Thus, we will classify all families of q-classical polynomials using the zeros of ϕ and $\phi^\star$ as it was suggested in [17,19].

In such a case as $\phi(0)=0$ if and only if $\phi^\star(0)=0$, in a first step, it is natural to classify the q-classical polynomials into two wide groups: the $\emptyset$-families, i.e., the families such that $\phi(0)\neq 0$

and the 0-families, i.e., the ones with $\phi(0)=0$. Next, we classify each member in the aforesaid two wide classes in terms of the degree of the polynomials ϕ and $\phi^\star$ as well as the multiplicity of their zeros in the case of 0-families. In fact, if ϕ has two simple zeros, the polynomials belong to the 0-Jacobi/–family while if the zeros are multiple, then they are a 0-Bessel/–family. So, we have the following schema for the q-classical OPS [18]:

$$\emptyset\text{-families}\begin{cases}\emptyset\text{-Jacobi/Jacobi}\\ \emptyset\text{-Jacobi/Laguerre}\\ \emptyset\text{-Jacobi/Hermite}\\ \\ \emptyset\text{-Laguerre/Jacobi}\\ \\ \emptyset\text{-Hermite/Jacobi}\end{cases}\qquad 0\text{-families}\begin{cases}0\text{-Bessel/Jacobi}\\ 0\text{-Bessel/Laguerre}\\ \\ 0\text{-Jacobi/Jacobi}\\ 0\text{-Jacobi/Laguerre}\\ 0\text{-Jacobi/Bessel}\\ \\ 0\text{-Laguerre/Jacobi}\\ 0\text{-Laguerre/Bessel}\end{cases}$$

Notice that in this schema the families ∅-Laguerre/Laguerre, ∅-Laguerre/Hermite ∅-Hermite/Laguerre and ∅-Hermite/Hermite cannot appear because of the connection between ϕ and $\phi^\star$, as well as the 0-Bessel/Bessel case, since they do not correspond to a quasi-definite functional (see Proposition 2.7).

3.2.1. Connection with the Nikiforov–Uvarov and the q-Askey tableaus

Here we will identify our classification (schema) of the q-classical polynomials with two well known ones in [23] and the q-Askey tableau [13]. Furthermore, we will identify all polynomials on the exponential lattice that appeared in the Nikiforov–Uvarov tableau with the q-classical ones, as we show that the polynomials given on the Askey tableau correspond to a particular choice of the parameters. Finally, we will show that in the q-Askey tableau two families of q-polynomials are missing.

We start with the first one. The Nikiforov–Uvarov tableau is based on the polynomial solutions of the *second-order linear difference equation of hypergeometric type in the nonuniform lattice* $x(s)$:

$$\tilde{\sigma}(x(s))\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla y_n[x(s)]}{\nabla x(s)}+\frac{\tilde{\tau}(x(s))}{2}\left[\frac{\Delta y_n[x(s)]}{\Delta x(s)}+\frac{\nabla y_n[x(s)]}{\nabla x(s)}\right]+\lambda y_n[x(s)]=0,$$

$$\nabla f(s)=f(s)-f(s-1),\qquad \Delta f(s)=f(s+1)-f(s),\quad y_n[x(s)]\in\mathbb{P}[x(s)]$$

$$x(s)=c_1(q)q^s+c_2(q)q^{-s}+c_3(q),\quad q\in\mathbb{C},\tag{3.1}$$

where $\tilde{\sigma}(x)$ and $\tilde{\tau}(x)$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and λ_n is a constant, or, equivalently

$$\sigma(s)\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla y_n[x(s)]}{\nabla x(s)}+\tau(s)\frac{\Delta y_n[x(s)]}{\Delta x(s)}+\lambda_n y_n[x(s)]=0,$$

$$\sigma(s)=\tilde{\sigma}(x(s))-\tfrac{1}{2}\tilde{\tau}(x(s))\Delta x(s-\tfrac{1}{2}),\quad \tau(s)=\tilde{\tau}(x(s)).\tag{3.2}$$

Here $\mathbb{P}[x(s)]$ denotes the linear space of polynomials in $x(s)$. Notice that, if $x(s)=c_1q^s\equiv x$, i.e., we are in the so-called *linear exponential lattice*, then

$$\frac{\Delta y_n[x(s)]}{\Delta x(s)}=\Theta y_n(x) \quad \text{and} \quad \frac{\nabla y_n[x(s)]}{\nabla x(s)}=\Theta^\star y_n(x), \quad y_n(x)\equiv y_n[x(s)].$$

Thus, using the fact that $\Delta x(s-\frac{1}{2})=q^{-1/2}\Delta x(s)$, the hypergeometric equation (3.2) in the linear lattice $x(s)=c_1q^s$ can be rewritten as

$$\sigma(s)\Theta\Theta^\star y_n(x)+q^{-1/2}\tau(s)\Theta y_n(x)=-\lambda_n q^{-1/2}y_n(x), \quad y_n(x)\in\mathbb{P}.$$

Using the identity $\Theta=x(q-1)\Theta\Theta^\star+\Theta^\star$ we get

$$[\sigma+q^{-1/2}\tau(s)x(q-1)]\Theta\Theta^\star y_n(x)+q^{-1/2}\tau(s)\Theta^\star y_n(x)=-\lambda_n q^{-1/2}y_n(x),$$

which is nothing else than the $q-\mathscr{SL}$ equation (1.6) where

$$\sigma(s)=\phi+x(1-q)\psi=q\phi^\star, \qquad \tau(s)=q^{1/2}\psi, \quad \lambda_n=-q^{1/2}\hat{\lambda}_n. \tag{3.3}$$

In other words, the $q-\mathscr{SL}$ equation (1.6) is a second-order linear difference equation of hypergeometric type in the linear exponential lattice $x(s)=c_1q^s$. The above connection allows us to identify all the q-classical orthogonal polynomials (in the widespread Hahn's sense) with the q-polynomials in the exponential lattice in the Nikiforov et al. approach. In fact, using the explicit expression of the polynomials $\sigma(s)$ and $\sigma(s)+\tau(s)\Delta(x-\frac{1}{2})$ in the exponential lattice [23, Eqs. (84)–(85), p. 241 and Table page 244], we can identify our 12 classes of q-polynomials with the ones given in [23] (see Table 1). In order to identify the q-classical polynomials with the ones given in the q-Askey tableau [13] we rewrite the $q-\mathscr{SL}$ equation (1.6) in the following form:

$$\phi\cdot HP_n-(\phi+q^2\phi^\star)P_n+q^2\phi^\star\cdot H^{-1}P_n=(q-1)^2x^2\hat{\lambda}_nP_n. \tag{3.4}$$

Then, a simple comparison of the above difference equation with those given in the q-Askey tableau allows us to identify some of the families of the q-polynomials given in [13] with the corresponding q-classical ones, and so, with the ones in the Nikiforov–Uvarov tableau. This will be given in Table 1.

From Table 1 we see that the 0-Jacobi/Bessel and 0-Laguerre/Bessel families lead to new families of orthogonal polynomials. The reason why they do not appear in the q-Askey tableau will be considered latter on. Notice also that the class No. 8 from the Nikiforov–Uvarov tableau [23, p. 244] does not lead to any orthogonal polynomial sequence even in the widespread sense considered here.

3.3. The Rodrigues formula and hypergeometric representation

For the sake of completeness we will include here the identification of the q-classical polynomials in terms of the basic hypergeometric series [10]

$${}_r\varphi_p\left(\begin{matrix}a_1,a_2,\dots,a_r\\ b_1,b_2,\dots,b_p\end{matrix}\,\middle|\, q;\ z\right)=\sum_{k=0}^{\infty}\frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_p;q)_k}\frac{z^k}{(q;q)_k}[(-1)^kq^{\frac{k(k-1)}{2}}]^{p-r+1}, \tag{3.5}$$

where, as before, $(a;q)_k=\prod_{m=0}^{k-1}(1-aq^m)$.

Table 1
Comparison of the Nikiforov–Uvarov, the q-Askey and the q-classical polynomial tableaus

q-classical family		Nikiforov–Uvarov tableau [23]		q-Askey tableau [13]
$\emptyset$-Jacobi/Jacobi	$\Leftrightarrow$	Eq. (86) p. 242	$\Rightarrow$	The big q-Jacobi q-Hahn
$\emptyset$-Jacobi/Laguerre	$\Leftrightarrow$	No. 6 [23, p. 244]	$\Rightarrow$	q-Meixner quantum q-Kravchuk
$\emptyset$-Jacobi/Hermite	$\Leftrightarrow$	No. 12 [23, p. 244]	$\Rightarrow$	Al-Salam-Carlitz II discrete q^{-1}-Hermite II
$\emptyset$-Laguerre/Jacobi	$\Leftrightarrow$	No. 1 [23, p. 244]	$\Rightarrow$	Big q-Laguerre affine q-Kravchuk
$\emptyset$-Hermite/Jacobi	$\Leftrightarrow$	No. 2 [23, p. 244]	$\Rightarrow$	Al-Salam-Carlitz I discrete q-Hermite
0-Bessel/Jacobi	$\Leftrightarrow$	No. 4 [23, p. 244]	$\Rightarrow$	Alternative q-Charlier
0-Bessel/Laguerre	$\Leftrightarrow$	No. 11 [23, p. 244]	$\Rightarrow$	Stieltjes-Wigert
0-Jacobi/Jacobi	$\Leftrightarrow$	No. 3 [23, p. 244]	$\Rightarrow$	The little q-Jacobi q-Kravchuk
0-Jacobi/Laguerre	$\Leftrightarrow$	No. 10 [23, p. 244]	$\Rightarrow$	q-Laguerre q-Charlier
0-Jacobi/Bessel	$\Leftrightarrow$	No. 7 [23, p. 244]	$\Rightarrow$	New OP family
0-Laguerre/Jacobi	$\Leftrightarrow$	No. 5 [23, p. 244]	$\Rightarrow$	Little q-Laguerre (Wall)
0-Laguerre/Bessel	$\Leftrightarrow$	No. 9 [23, p. 244]	$\Rightarrow$	New OP family
—		No. 8 [23, p. 244]		—

3.3.1. The Rodrigues formula

Let us first obtain the "standard" Rodrigues formula.

Proposition 3.3. *Let $\boldsymbol{u} \in \mathbb{P}^*$ be a q-classical quasi-definite functional*, $(P_n)_{n \geqslant 0} = \operatorname{mops} \boldsymbol{u}$, *and ω the q-weight function defined by the q-Pearson equation* (2.1). *Then*,

$$P_n = q^{-n} r_n \frac{\Theta^{\star n}(H^{(n)}\omega)}{\omega}. \tag{3.6}$$

Proof. The proof of this proposition is straightforward. In fact, using the definition of $\omega^{(k)}$ we obtain $\omega^{(k)} = \phi^{(k-1)}\omega^{(k-1)}$. Thus, from (2.4) we have, for all $n \geqslant 1$,

$$\Theta^{\star n}(H^{(n)}\omega) \;=\; \Theta^{\star n}(\omega^{(n)}Q_0^{(n)}) = \frac{1}{[1]}\Theta^{\star n-1}[\Theta^{\star}(\phi^{(n-1)}\omega^{(n-1)}\Theta Q_1^{(n-1)}]$$

$$\stackrel{(2.4)}{=} \frac{q\hat{\lambda}_1^{(n-1)}}{[1]}\Theta^{\star n-1}[\omega^{(n-1)}Q_1^{(n-1)}] = \cdots = q^n \frac{\hat{\lambda}_1^{(n-1)}\cdots\hat{\lambda}_n}{[1]\cdots[n]}\omega P_n.$$

Finally, using expression (1.12) for the coefficient r_n the result follows. $\square$

The Rodrigues formula is very useful for finding the explicit expression of the polynomials P_n. In fact, using

$$\Theta^{\star n} f(x) = \frac{q^{\binom{n}{2}+n}}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k q^{k(k+1)/2-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q f(q^{k-n}x), \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

where $\binom{n}{2} = n(n-1)/2$, we get

$$P_n = \frac{q^{\binom{n}{2}} r_n}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k q^{k(k+1)/2-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{H^{(n)}\phi(xq^{k-n})\omega(q^{n-k}x)}{\omega(x)},$$

or, equivalently,

$$P_n = \frac{r_n(-1)^n}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k q^{\binom{n}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{H^{(n)}\phi(xq^{-k})\omega(q^{-k}x)}{\omega(x)}.$$

Now, taking into account the q-Pearson equation (2.6)

$$\frac{H\omega}{\omega} = \frac{\phi}{qH\phi^\star} \iff \frac{H^{-1}\omega}{\omega} = \frac{q\phi^\star}{H^{-1}\phi},$$

we obtain the following explicit expression for the q-classical polynomials in terms of the polynomials ϕ and $\phi^\star$:

$$P_n = \frac{r_n(-1)^n}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k q^{\binom{n}{2}+k} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=0}^{k-1} \phi^\star(xq^{-i}) \prod_{i=0}^{n-k-1} \phi(xq^i). \tag{3.7}$$

This formula is equivalent to the one obtained in [5, Eq. (4.14); 23, Eq. (33); 2, Eq. (2.24)] for the q-polynomials in the non-uniform lattice $x(s) = c_1 q^s + c_3$.

3.3.2. The hypergeometric representation

We start with the $\emptyset$-Jacobi/Jacobi family, i.e., the case where $\phi = \hat{a}(x-a_1)(x-a_2)$ and $\phi^\star = \hat{a}^\star(x-a_1^\star)(x-a_2^\star)$, $\hat{a}^\star a_1 a_2 \hat{a}^\star a_1^\star a_2^\star \neq 0$. The other cases can be obtained in a similar way. Then, substituting in the above expression we find for the q-classical polynomials the following representation:

$$P_n = \frac{r_n(\hat{a}a_1a_2)^n (x/a_1;q)_n (x/a_2;q)_n}{(1-q)^n x^n} \, {}_3\varphi_2 \left(\begin{matrix} q^{-n}, a_1^\star x^{-1}, a_2^\star x^{-1} \\ q^{1-n}a_1x^{-1}, q^{1-n}a_2x^{-1} \end{matrix} \,\middle|\, q;\ \frac{\hat{a}^\star}{\hat{a}} q^{-n+3} \right).$$

From the last formula it is not easy to see that P_n are polynomials on x of degree exactly equal to n. Thus, we will apply to the above equation transformations (3.2.5) and (3.2.3) given in [10, p. 61]. Notice that we can apply the transformation formula (3.2.5) [10, p. 61] because the polynomials ϕ and $q\phi^\star$ have the same independent term, and then the condition $\hat{a}a_1a_2 = q\hat{a}^\star a_1^\star a_2^\star$ is fulfilled. So, the hypergeometric representation of the monic q-classical $\emptyset$-Jacobi/Jacobi polynomials is

$$P_n(x) = \frac{a_2^n (a_1^\star/a_2;q)_n (a_2^\star/a_2;q)_n}{(a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1};q)_n} \, {}_3\varphi_2 \left(\begin{matrix} q^{-n}, a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1}, x/a_2 \\ a_1^\star/a_2, a_2^\star/a_2 \end{matrix} \,\middle|\, q;\ q \right). \tag{3.8}$$

Notice that, since ϕ and $\phi^\star$ are invariant with respect to the change $a_1 \leftrightarrow a_2$ and $a_1^\star \leftrightarrow a_2^\star$, then we can obtain an equivalent hypergeometric representation

$$P_n(x) = \frac{a_2^n (a_1^\star/a_1;q)_n (a_2^\star/a_1;q)_n}{(a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1};q)_n} \, {}_3\varphi_2 \left(\begin{matrix} q^{-n}, a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1}, x/a_1 \\ a_1^\star/a_1, a_2^\star/a_1 \end{matrix} \middle| \, q; \, q \right). \tag{3.9}$$

Notice also that from any of the above two formulas follows that P_n is a polynomial of degree n. Before starting a detailed study of each case let us write another equivalent form for the $\emptyset$-Jacobi/Jacobi polynomials which can be obtained applying to (3.8) transformation (III.12) from [10, p. 241–242]

$$P_n(x) = \frac{q^{\binom{n}{2}} (-a_2^\star)^n (a_1^\star/a_2;q)_n (a_1^\star/a_1;q)_n}{(a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1};q)_n} \, {}_3\varphi_2 \left(\begin{matrix} q^{-n}, a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1}, a_1^\star/x \\ a_1^\star/a_2, a_1^\star/a_1 \end{matrix} \middle| \, q; \, qx/a_2^\star \right). \tag{3.10}$$

If we now choose $\phi = aq(x-1)(bx-c)$ and $\phi^\star = q^{-2}(x-aq)(x-cq)$, then Theorem 2.6 and Eq. (3.8) give, for the weight function and the polynomials, respectively:

$$\omega(x) = \frac{(x/a, x/c;q)_\infty}{(bx/c, x;q)_\infty}, \qquad p_n(x;a,b,c;q) = \frac{(aq;q)_n (cq;q)_n}{(abq^{n+1};q)_n} \, {}_3\varphi_2 \left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \middle| \, q; q \right),$$

i.e., the big q-Jacobi polynomials. If we now choose $c = q^{-N-1}$ they become the q-Hahn polynomials $Q_n(x;a,b,N|q)$ (usually they are written as polynomials in $x = q^{-s}$, see [13,18]). Obviously, if we use formulas (3.9) and (3.10) instead of formula (3.8) we obtain other representations for the big q-Jacobi polynomials.

For the reminder of the cases we can do the same, substitute the polynomials ϕ and $\phi^\star$ in (3.7) and make the corresponding calculations, but here we will show how, from the q-classical $\emptyset$-Jacobi/Jacobi polynomials, all other cases can be derived by taking the appropriate limits, in fact by using the different confluent processes in the q-difference equation (3.4). A similar study only of polynomial solutions, not necessarily orthogonal, has been done in [23]. Here we will extend it to orthogonal q-orthogonal polynomials. We will give the details only in some special "difficult" cases or when clarity and accuracy are required.

We continue with the q-classical $\emptyset$-Jacobi/Laguerre polynomials. To obtain them we take the limit $a_2^\star \to \infty$. Then, $\phi = \hat{a}(x-a_1)(x-a_2)$ and

$$\begin{aligned} q\phi^\star &= q\hat{a}^\star (x-a_1^\star)(x-a_2^\star) = q\hat{a}^\star a_2^\star (x-a_1^\star)(x/a_2^\star - 1) \\ &= \frac{\hat{a} a_1 a_2}{a_1^\star}(x-a_1^\star)(x/a_2^\star - 1) \to -\frac{\hat{a} a_1 a_2}{a_1^\star (x-a_1^\star)}, \end{aligned}$$

where the relation $\hat{a} a_1 a_2 = q\hat{a}^\star a_1^\star a_2^\star$ has been used. In this case and since

$$\lim_{a_2^\star \to \infty} \frac{(a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1};q)_k}{(a_2^\star/a_2;q)_k} = q^{(n-1)k} \left(\frac{a_1^\star}{a_1} \right)^k,$$

Eq. (3.8) becomes

$$P_n(x) = \left(\frac{a_1 a_2}{a_1^\star} \right)^n (a_1^\star/a_2;q)_n q^{-n(n-1)} \, {}_2\varphi_1 \left(\begin{matrix} q^{-n}, x/a_2 \\ a_1^\star/a_2 \end{matrix} \middle| \, q; \, q^n a_1^\star/a_1 \right). \tag{3.11}$$

If we choose now $\phi=(x-1)(x+bc)$ and $\phi^\star=q^{-2}c(x-bq)$, then we obtain the q-Meixner polynomials

$$M_n(x;b,c;q)=(-c)^n(bq;q)_nq^{-n^2}\,{}_2\varphi_1\left(\begin{matrix}q^{-n},x\\ bq\end{matrix}\,\middle|\,q;-\frac{q^{n+1}}{c}\right).$$

In this case $\omega(x)=(x/b;q)_\infty/(-x/bc,x;q)_\infty$. Putting in the above formulas $b=q^{-N-1}$ and $c=-p^{-1}$ we obtain the Quantum q-Kravchuk polynomials $K_n^{qtm}(x;p,N;q)$.

The next family is the q-classical $\emptyset$-Jacobi/Hermite one. In this case we take the limit $a_1^\star,a_2^\star\to\infty$. Then, $\phi=\hat{a}(x-a_1)(x-a_2)$ and $q\phi^\star=\hat{a}^\star a_1a_2$. Thus (3.8) becomes

$$P_n(x)=(-a_2)^{-n}q^{\binom{n}{2}}\,{}_2\varphi_0\left(\begin{matrix}q^{-n},x/a_2\\ —\end{matrix}\,\middle|\,q;\,q^na_2/a_1\right). \tag{3.12}$$

Choosing $\phi=(x-a)(x-1)$ and $q\phi^\star=a$ we obtain the Al-Salam and Carlitz II polynomials

$$V_n^{(a)}(x;q)=(-a)^nq^{-\binom{n}{2}}\,{}_2\varphi_0\left(\begin{matrix}q^{-n},x\\ 0\end{matrix}\,\middle|\,q;\frac{q^n}{a}\right).$$

If now $\phi=(x-\mathrm{i})(x+\mathrm{i})$ and $q\phi^\star=1$, we obtain the discrete q-Hermite polynomials II $\tilde{h}_n(x;q)$

$$\tilde{h}_n(x;q)=\mathrm{i}^{-n}\,{}_2\varphi_0\left(\begin{matrix}q^{-n},ix\\ —\end{matrix}\,\middle|\,q;-q^{-n}\right)=x^n\,{}_2\varphi_1\left(\begin{matrix}q^{-n},q^{-n+1}\\ 0\end{matrix}\,\middle|\,q^2;-\frac{q^2}{x^2}\right),$$

and for the weight function we have $\omega(x)=(\mathrm{i}x,-\mathrm{i}x;q)_\infty^{-1}=(-x^2;q^2)_\infty^{-1}=(\prod_{k=0}^\infty(1+x^2q^{2k}))^{-1}$.

The q-classical $\emptyset$-Laguerre/Jacobi polynomials. In this case $a_2\to\infty$. Then, $\phi=-q\hat{a}^\star a_1^\star a_2^\star a_1^{-1}(x-a_1)$ and $\phi^\star=\hat{a}^\star(x-a_1^\star)(x-a_2^\star)$, thus from Eq. (3.10)

$$\begin{aligned}P_n(x)&=(-a_2^\star)^nq^{\binom{n}{2}}(a_1^\star/a_1;q)_n\,{}_2\varphi_1\left(\begin{matrix}q^{-n},a_1^\star/x\\ a_1^\star/a_1\end{matrix}\,\middle|\,q;\,qx/a_2^\star\right)\\ &=a_1^n(a_1^\star/a_1;q)_n(a_2^\star/a_1;q)_n\,{}_3\varphi_2\left(\begin{matrix}q^{-n},x/a_1,0\\ a_1^\star/a_1,a_2^\star/a_1\end{matrix}\,\middle|\,q;\,q\right).\end{aligned} \tag{3.13}$$

The last equality follows from the Jackson transformation formula (see [10, Eq. (III.5), p. 241]), or, directly, taking the limit in formula (3.9). If we now choose $\phi=-acq(x-1)$ and $\phi^\star=q^{-2}(x-aq)(x-cq)$, we obtain the big q-Laguerre polynomials

$$\begin{aligned}p_n(x;a,c;q)&=(aq;q)_n(cq;q)_n\,{}_3\varphi_2\left(\begin{matrix}q^{-n},0,x\\ aq,cq\end{matrix}\,\middle|\,q;q\right)\\ &=(aq;q)_n(-cq)^nq^{\binom{n}{2}}\,{}_2\varphi_1\left(\begin{matrix}q^{-n},aqx^{-1}\\ aq\end{matrix}\,\middle|\,q;\frac{x}{c}\right).\end{aligned}$$

Notice that they are nothing else than the big q-Jacobi polynomials when $b=0$. Here $\omega(x)=(x/a,x/c;q)_\infty/(x;q)_\infty$. To this class also belong the $K_n^{\mathrm{aff}}(x;p,N;q)$. In fact they are big q-Laguerre polynomials with parameters $a=q^{-N-1}$ and $c=p$.

The q-classical $\emptyset$-Hermite/Jacobi polynomials. In this case $a_1, a_2 \to \infty$, thus $\phi = q\hat{a}a_1^\star a_2^\star$ and $\phi^\star = \hat{a}^\star(x - a_1^\star)(x - a_2^\star)$. Then, from Eq. (3.10) one easily finds

$$P_n(x) = q^{\binom{n}{2}}(-a_2^\star)^n\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, a_1^\star/x \\ 0 \end{matrix}\,\middle|\, q;\, qx/a_2^\star\right). \tag{3.14}$$

Now choosing $\phi = a$ and $q\phi^\star = (x-1)(x-a)$, (3.14) leads to the Al-Salam and Carlitz I polynomials

$$U_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix}\,\middle|\, q;\, \frac{xq}{a}\right).$$

In this case the q-weight function is $\omega(x) = (qx/a, qx; q)_\infty$. If we put $a = -1$, then the Al-Salam and Carlitz I polynomials become the discrete q-Hermite polynomials I $h_n(x;q)$.

For the 0-families the situation is more complicated and a new parameter δ should be included.

To obtain the 0-Bessel/Jacobi polynomials we will take the limit $a_1, a_2, a_2^\star \to 0$. Thus, $\phi = \hat{a}x^2$ and $\phi^\star = \hat{a}^\star(x - a_1^\star)x$, but now we have a problem taking the limit in the expression $(a_1^\star a_2^\star a_1^{-1} a_2^{-1} q^{n-1}; q)_k$, so we will force the parameters $a_1, a_2, a_2^\star$ to tend to zero such that $a_2^\star a_1^{-1} a_2^{-1} = q^\delta$, with δ a fixed constant such that $q^\delta = \hat{a}/(q\hat{a}^\star a_1^\star)$. Then, taking the limit in Eq. (3.8) we obtain

$$P_n(x) = \frac{q^{\binom{n}{2}}(-a_1^\star)^n}{(q^{n+\delta-1}; q)_n}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, q^{n+\delta-1} \\ 0 \end{matrix}\,\middle|\, q;\, qx/a_1^\star\right), \qquad q^\delta = \frac{\hat{a}}{q\hat{a}^\star a_1^\star}. \tag{3.15}$$

To this class belong the Alternative q-Charlier polynomials $K_n(x;a,q)$. In fact, putting $\phi = ax^2$ and $\phi^\star = q^{-2}x(1-x)$, thus $q^\delta = -aq$ and then

$$K_n(x;a;q) = \frac{(-1)^n q^{\binom{n}{2}}}{(-aq^n; q)_n}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, -aq^n \\ 0 \end{matrix}\,\middle|\, q; qx\right).$$

For them we have $\omega(x) = |x|^\alpha (x^{-1}; q)_\infty^{-1}$, where $q^\alpha = -a/q$.

For the 0-Bessel/Laguerre polynomials we have the limit $a_1, a_2, a_1^\star \to 0$ and $a_2^\star \to \infty$. Thus, $\phi = \hat{a}x^2$ and $\phi^\star = \hat{a}^\star(x - a_1^\star)(x - a_2^\star) = \hat{a}^\star a_2^\star(x/a_2^\star - 1)(x - a_1^\star) = \hat{a}a_1a_2a_1^{\star -1}q^{-1}(x/a_2^\star - 1)(x - a_1^\star)$. If we now take the limit in such a way that $a_1^\star a_1 a_2 = -q^\delta$ we get the function $\phi^\star = \hat{a}q^{-\delta-1}x$. In this case Eq. (3.10) immediately gives

$$P_n(x) = q^{-n(n+\delta-1)}(-1)^n\, {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix}\,\middle|\, q; -q^{n+\delta}x\right), \qquad q^\delta = -\frac{\hat{a}}{\bar{a}^\star q}. \tag{3.16}$$

Now, setting $\phi = x^2$ and $\phi^\star = q^{-2}x$, we have $q^\delta = -q$ and we obtain the Stieltjes–Wigert polynomials

$$S_n(x;q) = (-1)^n q^{-n^2}\, {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix}\,\middle|\, q; -xq^{n+1}\right).$$

Here $\omega(x) = \sqrt{x^{\log_q x - 1}}$.

The 0-Jacobi/Jacobi polynomials. In this case the limit is $a_2, a_2^\star \to 0$ assuming that $a_2^\star/a_2 = q^\delta$. Then $\phi = \hat{a}x(x-a_1)$, $\phi^\star = \hat{a}^\star x(x-a_1^\star)$ and (3.8) gives

$$P_n(x) = \frac{q^{\binom{n}{2}}(-a_1^\star)^n(q^\delta;q)_n}{(a_1^\star/a_1 q^{\delta+n-1};q)_n}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, a_1^\star/a_1 q^{n+\delta-1} \\ q^\delta \end{matrix}\,\middle|\, q; qx/a_1^\star\right), \qquad q^\delta = \frac{\hat{a}a_1}{q\hat{a}^\star a_1^\star}. \tag{3.17}$$

Putting $\phi = ax(bqx-1)$ and $\phi^\star = q^{-2}x(x-1)$, $q^\delta = aq$, thus

$$p_n(x;a,b|q) = \frac{(-1)^n q^{\binom{n}{2}}(aq;q)_n}{(abq^{n+1};q)_n}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}\,\middle|\, q; qx\right),$$

which are nothing else than the Little q-Jacobi polynomials. If now we take $\phi = px(1-x)$, $\phi^\star = q^{-2}x(x-q^{-N})$ we get

$$K_n(x;p,N;q) = \frac{(-1)^n q^{-nN+\binom{n}{2}}(-pq^{N+1};q)_n}{(-pq^n;q)_n}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, -pq^n \\ -pq^{N+1} \end{matrix}\,\middle|\, q; xq^{N+1}\right),$$

which constitutes an alternative definition for the q-Kravchuk polynomials equivalent to the "more" standard one just using the transformation formula (III.7) from [10, p. 241]

$$K_n(x;p,N;q) = \frac{(q^{-N};q)_n}{(-pq^n;q)_n}\, {}_3\varphi_2\left(\begin{matrix} q^{-n}, x, -pq^n \\ q^{-N}, 0 \end{matrix}\,\middle|\, q; q\right).$$

Finally, we have $\omega(x) = |x|^\alpha (qx;q)_\infty/(qbx;q)_\infty$, $q^\alpha = a$ and $\omega(x) = |x|^\alpha (q^{N+1}x;q)_\infty/(x;q)_\infty$, $q^\alpha = pq^N$ for the weight functions of the Little q-Jacobi and q-Kravchuk polynomials, respectively.

The 0-Jacobi/Laguerre polynomials. In this case we take the limit $a_2, a_2^\star \to 0$ and $a_1^\star \to \infty$ in such a way that $a_2^\star/a_2 = -q^\delta$, so $\phi = \hat{a}x(x-a_1)$, $\phi^\star = \hat{a}a_1 q^{-\delta-1}x = \bar{a}^\star x$, and then

$$P_n(x) = (-a_1)^n q^{-n(n+\delta-1)}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, x/a_1 \\ 0 \end{matrix}\,\middle|\, q; -q^{n+\delta}\right), \qquad q^\delta = \frac{\hat{a}a_1}{q\bar{a}^\star}. \tag{3.18}$$

Putting $\phi = ax(x+1)$ and $\phi^\star = q^{-2}x$, then $q^\delta = -aq$, and we obtain the q-Laguerre polynomials $L_n^\alpha(x;q) \equiv L_n(x;a;q)$

$$L_n(x;a;q) = (-1)^n q^{-n^2} a^{-n}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, -x \\ 0 \end{matrix}\,\middle|\, q; aq^{n+1}\right), \qquad \omega(x) = \frac{|x|^\alpha}{(-x;q)_\infty}, \quad q^\alpha = -a.$$

If we now choose $\phi = x(x-1)$ and $\phi^\star = q^{-2}ax$, then $q^\delta = q/a$ and we get the q-Charlier polynomials

$$C_n(x;a;q) = (-1)^n q^{-n^2} a^n\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, x \\ 0 \end{matrix}\,\middle|\, q; -\frac{q^{n+1}}{a}\right), \qquad \omega(x) = \frac{|x|^\alpha}{(x;q)_\infty}, \quad q^\alpha = a^{-1}.$$

The 0-Jacobi/Bessel polynomials. Here we take the limit $a_2, a_1^\star, a_2^\star \to 0$ in such a way that $a_1^\star a_2^\star / a_2 = q^\delta$, so $\phi = \hat{a}x(x - a_1)$, $\phi^\star = \hat{a}^\star x^2 = \hat{a}a_1 q^{-\delta-1}x^2$, and then (3.8) gives

$$P_n(x) = q^{n(n+\delta-1)}(q^{n+\delta-1}/a_1;q)_n^{-1}\,{}_2\varphi_0\left(\begin{matrix} q^{-n}, q^{n+\delta-1}/a_1 \\ — \end{matrix}\,\middle|\, q; xq^{1-\delta}\right), \qquad q^\delta = \frac{\hat{a}a_1}{q\hat{a}^\star}. \tag{3.19}$$

This family does not appear in the q-Askey Schema unless it is not trivial limit of a more general q-family. We will take the following parameterization $\phi = ax(x-b)$ and $\phi^\star = q^{-2}x^2$. Then, $q^\delta = abq$ and we obtain the 0-Jacobi/Bessel polynomials, denoted by $j_n(x;a,b)$

$$j_n(x;a,b) = (ab)^n q^{n^2}(aq^n;q)_n^{-1}\,{}_2\varphi_0\left(\begin{matrix} q^{-n}, aq^n \\ — \end{matrix}\,\middle|\, q; x/(ab)\right), \qquad \omega(x) = |x|^\alpha (bq/x;q)_\infty,$$

$$q^\alpha = a^{-1}q^{-5}.$$

Their main data are shown in Table 2.

The 0-Laguerre/Jacobi polynomials. In this case $a_2, a_2^\star \to 0$, $a_1 \to \infty$, $q^\delta = -a_2^\star/a_2$, then $\phi = \bar{a}x = \hat{a}^\star a_1^\star q^{\delta+1}x$, $\phi^\star = \hat{a}^\star x(x - a_1^\star)$, and

$$P_n(x) = (-a_1^\star)^n q^{\binom{n}{2}}(-q^\delta;q)_n\,{}_2\varphi_1\left(\begin{matrix} q^{-n}, 0 \\ -q^\delta \end{matrix}\,\middle|\, q; xq/a_1^\star\right), \qquad q^\delta = \frac{\bar{a}}{\hat{a}^\star a_1^\star q}. \tag{3.20}$$

Putting $\phi = -ax$ and $\phi^\star = q^{-2}x(x-1)$, thus $q^\delta = -aq$, and we obtain the Little q-Laguerre or Wall polynomials

$$p_n(x;a|q) = (-1)^n q^{\binom{n}{2}}(aq;q)_n\,{}_2\varphi_1\left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix}\,\middle|\, q; qx\right), \qquad \omega(x) = |x|^\alpha (qx;q)_\infty, \quad q^\alpha = -a.$$

Finally, the 0-Laguerre/Bessel family follows from Eq. (3.9) taking the limit $a_1, a_1^\star, a_2^\star \to 0$ and $a_2 \to \infty$ assuming that $a_1^\star a_2^\star / a_1 = -q^\delta$. Thus $\phi = \bar{a}x = \hat{a}^\star q^{\delta+1}x$, $\phi^\star = \hat{a}^\star x^2$ and

$$P_n(x) = (-1)^n q^{n(n+\delta-1)}\,{}_2\varphi_0\left(\begin{matrix} q^{-n}, 0 \\ — \end{matrix}\,\middle|\, q; -xq^{1-\delta}\right), \qquad q^\delta = \frac{\bar{a}}{q\hat{a}^\star}. \tag{3.21}$$

As in the case of 0-Jacobi/Bessel, a new family (which is not in the q-Askey tableau) appears. In this case we will adopt the parameterization $\phi = \bar{a}x = ax$, $\phi^\star = q^{-2}x^2$, $q^\delta = aq$, thus

$$P_n(x) \equiv l_n(x;a) = (-a)^n q^{n^2}\,{}_2\varphi_0\left(\begin{matrix} q^{-n}, 0 \\ — \end{matrix}\,\middle|\, q; -x/a\right), \qquad \omega(x) = |x|^\alpha \sqrt{x^{\log_q x^{-1}+1}}, \quad q^\alpha = a/q.$$

Remark 3.4. Notice that in some examples the q-weight functions look very different from the ones given in [13]. Sometimes the reason is that the associated moment problem is indeterminate (e.g., the Stieltjes–Wieger polynomials of the q-Laguerre polynomials). Also, because sometimes discrete sums are used instead of the q-integrals (see, e.g., the example of the Little q-Jacobi polynomials in [13]).

Notice that for all $0 < q < 1$, the polynomials $l_n(x;a)$ never constitute a positive definite family according to the Favard Theorem since $g_n < 0$. For the $j_n(x;a,b)$ polynomials the situation is

Table 2
The q-classical polynomials $j_n(x;a,b)$ and $l_n(x;a)$*

P_n	$j_n(x;a,b)$	$l_n(x;a)$
ϕ	$ax(x-b)$	ax
$\phi^\star$	$q^{-2}x^2$	$q^{-2}x^2$
ψ	$\dfrac{abq+(1-aq)x}{q(1-q)}$	$\dfrac{aq-x}{(q-1)q}$
λ_n	$-\dfrac{q^{-n}[n](1-aq^n)}{1-q}$	$\dfrac{q^{-n}[n]}{1-q}$
r_n	$\dfrac{q^{\binom{n}{2}+n}(1-q)^n}{(aq^n;q)_n}$	$q^{\binom{n}{2}+n}(1-q)^n$
d_n	$\dfrac{abq^n(1-q^n+aq^{2n}-q^{n+1})}{(1-aq^{2n-1})(1-aq^{2n+1})}$	$aq^n(q^n+q^{n+1}-1)$
g_n	$-\dfrac{a^2b^2q^{3n-1}(1-q^n)(1-aq^{n-1})}{(1-aq^{2n-1})^2(1-aq^{2n})(1-aq^{2n-2})}$	$a^2q^{3n-1}(q^n-1)$
a_n	$a[n]$	0
b_n	$-\dfrac{ab[n](1-aq^n)(1+aq^{2n})}{(1-aq^{2n-1})(1-aq^{2n+1})}$	$a[n]$
c_n	$\dfrac{a^2b^2q^{2n-1}[n](1-aq^n)(1-aq^{n-1})}{(1-aq^{2n-1})^2(1-aq^{2n})(1-aq^{2n-2})}$	$a^2q^{2n-1}[n]$
e_n	$\dfrac{abq^n(1-q^n)(1+aq^{2n})}{(1-aq^{2n-1})(1-aq^{1+2n})}$	$aq^n(q^n-1)$
h_n	$\dfrac{a^3b^2q^{4n-2}(1-q^n)(1-q^{n-1})}{(1-aq^{2n-1})^2(1-aq^{2n})(1-aq^{2n-2})}$	0
d'_n	$\dfrac{abq^{n+1}(1-q^n-q^{n+1}+aq^{2n+2})}{(1-aq^{2n+1})(1-aq^{2n+3})}$	$aq^{n+1}(q^n+q^{n+1}-1)$
g'_n	$-\dfrac{a^2b^2q^{3n+1}(1-q^n)(1-aq^{n+1})}{(1-aq^{2n})(1-aq^{2n+1})^2(1-aq^{2n+2})}$	$a^2q^{3n+1}(q^n-1)$

*Here d_n, g_n; a_n, b_n, c_n and e_n, h_n are the coefficients in formulas (1.1), (1.10), (1.11), respectively, as well as d'_n and g'_n are the coefficients in the recurrence relation for the monic derivatives $xQ_n = Q_{n+1} + d'_nQ_n + g'_nQ_{n-1}$.

more complicated. Nevertheless, choosing $a=q^{-N}$ it is easy to show that $j_n(x;a,b)$ constitutes a finite family (similar to the q-Hahn polynomials) which is positive definite since $g_n>0$ for all $n=0,1,\ldots,[N/2]$. The detailed study of the positive definite cases in terms of the zeros of ϕ and $\phi^\star$ will be considered in a forthcoming paper.

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